STAT 511 Homework 1

Due Date: 11:00 A.M., Wednesday, January 18

Please review Ken Koehler's notes on working with vectors and matrices available at

http://www.public.iastate.edu/∼kkoehler/stat511/sect2.4page.pdf

Here are a few points to note when examining the notes.

- We will use the notation $\boldsymbol{A}'$ to denote the transpose of the matrix $\boldsymbol{A}$ rather than $\boldsymbol{A}^T$ as in Dr. Koehler's notes.
- We will use R rather than S-PLUS as in Dr. Koehler's notes. However, Dr. Koehler's S-PLUS code should work fine in R.
- On slide 115, "frac13" should be $\frac{1}{3}$.

Once you have reviewed the notes, please answer the following questions. It is possible and preferable to complete the problems without the use of a computer. You are encouraged to check your work with the help of R.

1. Determine the rank of the following matrix.

$$\begin{bmatrix} 3 & 2 & 7 & 1 & 5 \\ -1 & 3 & 5 & 1 & 3 \\ 4 & 2 & 8 & 7 & -6 \end{bmatrix}$$

2. Consider the design matrix on slide 19 of the lecture notes available at

   http://www.public.iastate.edu/∼dnett/S511/01Introduction.pdf

   What is the largest possible rank of this matrix, and what must be true about $x_1, \ldots, x_8$ to achieve that maximum rank?

3. Consider the design matrix on page 30 of the lecture notes available at

   http://www.public.iastate.edu/∼dnett/S511/01Introduction.pdf

   (a) Determine the rank of design matrix.

   (b) The column space of a matrix is the set of vectors spanned by the columns of the matrix. Give another matrix with the same column space as the design matrix.

4. Consider the subset of $\mathbb{R}^3$ given by $S = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{1}'\boldsymbol{x} = 0\}$. (Here $\boldsymbol{1} = [1, 1, 1]'$.)

   (a) Show that S is a vector space.

   (b) Provide a basis for S.

   (c) State the dimension of S.

5. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are random vectors, $\text{Cov}(\boldsymbol{x}, \boldsymbol{y})$ is defined as $\text{E}[\{\boldsymbol{x} - \text{E}(\boldsymbol{x})\}\{\boldsymbol{y} - \text{E}(\boldsymbol{y})\}']$. (Note that it is not necessary for $\boldsymbol{x}$ and $\boldsymbol{y}$ to be of the same order.) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of constants, $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors of constants, and $\boldsymbol{z}$ is a random vector with mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\Sigma}$. Give a simplified expression for $\text{Cov}(\boldsymbol{A}\boldsymbol{z} + \boldsymbol{a}, \boldsymbol{B}\boldsymbol{z} + \boldsymbol{b})$.

6. Suppose $\boldsymbol{A}$ is a matrix. Prove that $\boldsymbol{A} = \boldsymbol{0}$ if and only if $\operatorname{tr}(\boldsymbol{A}'\boldsymbol{A}) = 0$.

7. Suppose $a_1, a_2, a_3, a_4 \in I\!R$. Consider the matrix

$$\mathrm{diag(a_1, a_2, a_3, a_4)} \equiv \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}.$$

   (a) Find the determinant of $\mathrm{diag(a_1, a_2, a_3, a_4)}$.

   (b) Find the eigenvalues of $\mathrm{diag(a_1, a_2, a_3, a_4)}$.

8. Result 1.10 (iii) is known as the cyclic property of the trace. Use this property to prove Result 1.10 (iv).

9. Prove Results 1.11 (iv), (vi), and (vii). You may use any results stated before slide 86 of Dr. Koehler's notes. You are not allowed to use results stated after slide 86 without first proving those results.

10. Result 1.12 is sometimes known as the Spectral Decomposition Theorem. This result is very useful and powerful. Use this result along with the cyclic property of the trace to prove Result 1.11 (v).

11. Find a so that

$$\begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix} \text{ is in the span of } \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ 8 \end{bmatrix} \right\}.$$

12. Suppose W_1 and W_2 are random variables. Suppose $\mathrm{Var(W_1)} = 4$, $\mathrm{Var(W_2)} = 2$, and $\mathrm{Cov(W_1, W_2)} = -1$. Find

$$\mathrm{Var}\left(\begin{bmatrix} W_1 + W_2 \\ W_1 - W_2 \end{bmatrix} \right).$$