1a) See slide 1 of our course notes.

b) \( x(x'x)^{-1}x'y \)

c) \( x'xb = x'y \)

d) Yes. \( x(x'x)^{-1}x'y = x(x'x)^{-1}x'xb \)
\[ = px'xb \]
\[ = x'b. \]

2. Let \( A \) be any symmetric and idempotent matrix. Then

\[ A = AA' = A(AA)' = A(A'A)' = PA. \]

Thus, \( A \) is the orthogonal projection matrix that projects onto \( \mathbb{C}(A) \).
b) A quantity is estimable if and only if it can be written as a linear combination of $E(y) = X\beta$, or, equivalently, if there is a linear function of $y$ whose expectation is equal to the quantity.

i) $\mu + S_1 = E(y_{12})$. Thus, $\mu + S_1$ is estimable.

ii) $\mu + S_1 + 10\beta = E(y_{12} + 10(y_{13} - y_{12})) = E(10y_{13} - 9y_{12})$.
Thus, $\mu + S_1 + 10\beta$ is estimable.
(continued)

iii) \( s_1 - s_2 = E(Y_{121} - Y_{221}) \).

Thus, \( s_1 - s_2 \) is estimable.

iv) In order for \( \mu \) to be estimable, we need to find \( \alpha \) so that \( \alpha'X\beta = \mu \), i.e.

\[
\alpha'X = [1, 0, 0, 0],
\]

\[
\alpha'X = [1, 0, 0, 0] \implies \sum_{m=1}^{12} a_m = 1, \text{ and }
\]

\[
\sum_{m=1}^{6} a_m = 0, \text{ and }
\]

\[
\sum_{m=7}^{12} a_m = 0.
\]

Because \( \sum_{m=1}^{6} a_m = 0 \) and \( \sum_{m=7}^{12} a_m = 0 \implies \sum_{m=1}^{12} a_m = 0 \), there does not exist \( \alpha \) such that \( \alpha'X\beta = \mu \). Therefore, \( \mu \) is not estimable.
There are many other ways to complete 3b). Using the method of HW 2, Problem 7 would work like this.

Suppose \( X d = 0 \). Then

\[
\begin{align*}
  d_1 + d_2 - d_4 &= 0 \\
  d_1 + d_2 &= 0 \\
  d_1 + d_2 + d_4 &= 0 \\
  d_1 + d_3 - d_4 &= 0 \\
  d_1 + d_3 &= 0 \\
  d_1 + d_3 + d_4 &= 0
\end{align*}
\]

which means that \( X d = 0 \) \( \iff \)

\( d \) is of the form \( [d, -d, -d, 0]' \), \( d \in \mathbb{R} \).
Now note that

\[ \begin{align*}
\text{i}) \quad M + S_1 &= [1, 1, 0, 0] \beta \\
[1, 1, 0, 0] \begin{bmatrix} d \\ -d \\ -d \\ 0 \end{bmatrix} &= 0 \quad \forall \ d \in \mathbb{R}.
\end{align*} \]

Thus, \( M + S_1 \) is estimable.

\[ \begin{align*}
\text{ii}) \quad M + S_1 + 10 \beta &= [1, 1, 0, 10] \beta \\
[1, 1, 0, 10] \begin{bmatrix} d \\ -d \\ -d \\ 0 \end{bmatrix} &= 0 \quad \forall \ d \in \mathbb{R}.
\end{align*} \]

Thus, \( M + S_1 + 10 \beta \) is estimable.

\[ \begin{align*}
\text{iii}) \quad S_1 - S_2 &= [0, 1, -1, 0] \beta \\
[0, 1, -1, 0] \begin{bmatrix} d \\ -d \\ -d \\ 0 \end{bmatrix} &= 0 \quad \forall \ d \in \mathbb{R}
\end{align*} \]

Thus, \( S_1 - S_2 \) is estimable.

\[ \begin{align*}
\text{iv}) \quad M &= [1, 0, 0, 0] \beta \\
[1, 0, 0, 0] \begin{bmatrix} d \\ -d \\ -d \\ 0 \end{bmatrix} &= d \neq 0 \quad \forall \ d \in \mathbb{R}.
\end{align*} \]

Thus, \( M \) is NOT estimable.
3c) The X matrix in 3(a) has rank 3. Any one of the first three columns can be written as a linear combination of the other two. R would discard the second column of the matrix, but calculations are much easier if columns of the model matrix are orthogonal. Thus, I will remove the first column and use X as the model matrix.

\[
X = \begin{bmatrix}
1 & 0 & -1 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

Our model specification in the problem statement implies that \( E(y) = \begin{bmatrix} M + S_1 - \beta \\ M + S_1 - \beta \\ M + S_1 \\ M + S_1 + \beta \\ M + S_2 - \beta \\ M + S_2 - \beta \\ M + S_2 \\ M + S_2 \end{bmatrix} \) implies that \( E(y) = \begin{bmatrix} M + S_1 - \beta \\ M + S_1 - \beta \\ M + S_1 \\ M + S_1 + \beta \\ M + S_2 - \beta \\ M + S_2 - \beta \\ M + S_2 \\ M + S_2 \end{bmatrix} \).

It is easy to see that my X matrix multiplied on the right by \( \begin{bmatrix} M + S_1 \\ M + S_2 \\ \beta \end{bmatrix} \) will give \( E(y) \).

Thus, \( (X'X)^{-1}X'y = \beta = \begin{bmatrix} \frac{M + S_1}{M + S_2} \\ \frac{M + S_2}{M + S_2} \\ \beta \end{bmatrix} \).
3 d) The BLUE of $E(Y_{111})$ is 
\[ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \hat{\beta}, \text{ where } \hat{\beta} = (X'X)^{-1}X'Y. \]

First row of $X$

\[ X'X = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix} \]

\[ (X'X)^{-1} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix} \]

\[ X'Y = \begin{bmatrix} Y_{1..} \\ Y_{2..} \\ Y_{3..} - Y_{1..} \end{bmatrix} \]

\[ (X'X)^{-1}X'Y = \begin{bmatrix} \overline{Y}_{1..} \\ \overline{Y}_{2..} \\ \frac{1}{2}(\overline{Y}_{3..} - \overline{Y}_{1..}) \end{bmatrix} \]

Therefore, \( E(Y_{111}) = \overline{Y}_{1..} - \frac{1}{2}(\overline{Y}_{3..} - \overline{Y}_{1..}) \).

3 e) \( \overline{Y}_{1..} + s_{1} \)

\( \overline{Y}_{1..} + s_{1} + 10\beta \) = \( \overline{Y}_{1..} + s (\overline{Y}_{3..} - \overline{Y}_{1..}) \)

\( s_{1} - s_{2} \) = \( \overline{Y}_{1..} - \overline{Y}_{2..} \)
4a) The F-test at the very bottom of the output tests whether all model coefficients other than the intercept are simultaneously equal to 0. Thus, this is a test of one mean for all observations vs. one mean for each combination of source and protein amount (the cell means model).

\[ F = 39.84 \]
\[ df = 5, 6 \]
\[ p-value = 0.0001587 \]

There is strong evidence of a difference among treatment means.
b) The design matrix used by R is

\[
\begin{pmatrix}
1 & 1 & x & x^2 & sx & sx^2 \\
10 & -1 & 1 & 0 & 0 \\
10 & -1 & 1 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 \\
10 & 1 & 1 & 0 & 0 \\
10 & 1 & 1 & 0 & 0 \\
11 & -1 & 1 & -1 & 1 \\
11 & -1 & 1 & -1 & 1 \\
11 & 0 & 0 & 0 & 0 \\
11 & 0 & 0 & 0 & 0 \\
11 & 1 & 1 & 1 & 1 \\
11 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

The mean for Source 2, Protein Amount 3 is given by either of the last two rows times \( \beta \). Thus, the estimate is the sum of \( \hat{\beta} = 17.5 + 5 + 6.25 - 0.25 + 3.75 + 0.75 \)

\[= 33\]
4b) This can also be seen by noting that the model is

\[ Y_{ijk} = \beta_0 + \beta_1 S(i) + \beta_2 X_j + \beta_3 X_j^2 + \beta_4 S(i) \times X_j + \beta_5 S(i) \times X_j^2 + \varepsilon_{ijk} \]

where \( S(i) = \begin{cases} 1 & \text{if } i = 2 \\ 0 & \text{if } i = 1 \end{cases} \) and \( X_j = j - 2 \).

Thus, for \( i = 2, j = 3 \), the mean is

\[ \beta_0 + \beta_1(1) + \beta_2(3-2) + \beta_3(3-2)^2 + \beta_4(1)(3-2) + \beta_5(1)(3-2)^2 \]

\[ = \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5. \]

C) The standard error of any cell mean is

\[ \sqrt{\text{Var}(\hat{Y}_{ij})} = \sqrt{\frac{\hat{\sigma}^2}{2}} = \sqrt{\frac{\text{MSE}}{2}} = \sqrt{\frac{3.25}{2}}. \]

Also, the estimated intercept is an estimate of \( M_{11} \), which has the same standard error as \( \hat{M}_{23} \). R gives 1.2748 as \( SE \) for intercept.
4d) From part (b), we can see that the average of the source 1 means is \[ \left[ 1, 0, 0, \frac{2}{3}, 0, 0 \right] \beta \]

\[ \downarrow \]

Average of 1st 6 rows of \( X \).

The average of the source 2 means is \[ \left[ 1, 1, 0, \frac{2}{3}, 0, \frac{2}{3} \right] \beta \].

\[ \downarrow \]

Average of the last 6 rows of \( X \).

Thus, \( C = \left[ 0, 1, 0, 0, 0, \frac{2}{3} \right] \) will do.

\[ \downarrow \]

Difference of vectors above.
4 e) \((19.5 + 0.37 + 28.13 + 0.04) / 9\)

4 f) \(F = \frac{(SSE_{\text{reduced}} - SSE_{\text{full}})}{(dFE_{\text{reduced}} - dFE_{\text{full}})} \cdot \frac{\text{MSE}_{\text{full}}}{\text{MSE}_{\text{reduced}}}

= \frac{(0.37 + 28.13 + 0.04)}{(9 - 6)}

= 3.25

\(dF = 3, 6\)
5. \( W \sim \chi^2_m (\delta^2) \Rightarrow W = (\overline{Z} + \delta)^T (\overline{Z} + \delta) \)

where \( \overline{Z} \sim N(0, I) \) and \( \delta = \begin{bmatrix} \delta_1 \\
\vdots \\
\delta_n \end{bmatrix} \).

\[ a) \quad E(W) = E(\overline{Z}^T \overline{Z} + 2 \overline{Z} \delta + \delta^T \delta) \]

\[ = E(\overline{Z}^T \overline{Z}) + 2 \delta^T E(\overline{Z}) + \delta^2 \]

\[ = E(\chi^2_m) + 2 \delta^T \overline{Q} + \delta^2 \]

\[ = m + \delta^2 \]

\[ b) \quad Var(W) = Var(\overline{Z}^T \overline{Z} + 2 \overline{Z} \delta + \delta^T \delta) \]

\[ = Var(\sum_{i=1}^{m} Z_i^2 + 2 \delta \overline{Z}) \]

\[ = Var(\sum_{i=1}^{m} Z_i^2) + 4 \delta^2 Var(Z_1) + 0 \]

(because \( \text{Cov}(Z_1^2 + \cdots + Z_m^2, Z_1) = \text{Cov}(Z_1^2, Z_1) = 0 \))

\[ = Var(\chi^2_m) + 4 \delta^2 = 2m + 4 \delta^2. \]
Point values were set as follows:

1. a) 4  
   b) 4  
   c) 4  
   d) 5

2. 6

3. a) 4  
   b) 4 x 4
   c) 4  
   d) 6  
   e) 6

4. a) 5  
   b) 5  
   c) 5  
   d) 5  
   e) 5  
   f) 6

5. a) 5  
   b) 5