For linear combinations of mean squares:

Suppose $MS_1, \ldots, MS_k$ are independent random variables and that

\[
S^2 = a_1 MS_1 + a_2 MS_2 + \ldots + a_k MS_k,
\]

where $a_1, a_2, \ldots, a_k$ are known constants in $\mathbb{R}$. Consider the random variable $\frac{df_i MS_i}{E(\text{MS}_i)} \sim X^2_{df_i}$, $i = 1, \ldots, k$.

Recall that \( \frac{df_i MS_i}{E(\text{MS}_i)} \sim \chi^2_{df_i} \).

If $S^2 = a_1 MS_1 + a_2 MS_2 + \ldots + a_k MS_k$ is distributed like each of the random variables in the linear combination, then

\[
\frac{df S^2}{E(S^2)} \sim \chi^2_{df}.
\]

Note that $E \left[ \frac{S^2}{E(S^2)} \right] = df = E(\chi^2_{df}) = 2df$. Equating this expression to $\text{Var}(\frac{S^2}{E(S^2)})$ yields

\[
\frac{\text{Var}(S^2)}{\left[ E(S^2) \right]^2} = \frac{df^2}{2 \left[ E(S^2) \right]^2} \text{Var}(S^2).
\]

A natural estimator of $\text{Var}(S^2)$ is

\[
\text{Var}(S^2) = 2 \sum_{i=1}^k a_i^2 \text{MS}_i^2 / df.
\]
Replacing $E(s^2)$ with $s^2$ and $\text{Var}(s^2)$ with $\hat{\text{Var}}(s^2)$ yields

$$\frac{(s^2)^2}{\sum_{i=1}^{k} a_i^2 MS_i / df_i} = \frac{(\sum_{i=1}^{k} a_i MS_i)^2}{\sum_{i=1}^{k} a_i^2 MS_i^2 / df_i},$$

which is the Cochran-Satterthwaite formula for the approximate degrees of freedom of a linear combination of mean squares.