THE AITKEN MODEL

- $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},\ \boldsymbol{\epsilon} \sim (\boldsymbol{0},\ \sigma^2\boldsymbol{V})$
- Identical to the Gauss-Markov linear model except that

$$\text{Var}(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{V} \text{ instead of } \sigma^2\boldsymbol{I}.$$

- $\boldsymbol{V}$ is assumed to be a known nonsingular variance matrix.
- σ^2 is an unknown positive variance parameter.

A Transformation of the Model

- By the Spectral Decomposition Theorem, there exists a nonsingular symmetric matrix $\boldsymbol{V}^{1/2}$ such that $\boldsymbol{V}^{1/2}\boldsymbol{V}^{1/2} = \boldsymbol{V}$.
- Using $\boldsymbol{V}^{-1/2}$ to denote $(\boldsymbol{V}^{1/2})^{-1}$, we have

$$\boldsymbol{V}^{-1/2}\boldsymbol{y} = \boldsymbol{V}^{-1/2}\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{V}^{-1/2}\boldsymbol{\epsilon}.$$

- With $\boldsymbol{z} = \boldsymbol{V}^{-1/2}\boldsymbol{y},\ \boldsymbol{W} = \boldsymbol{V}^{-1/2}\boldsymbol{X}$, and $\boldsymbol{\delta} = \boldsymbol{V}^{-1/2}\boldsymbol{\epsilon}$, we have

$$\boldsymbol{z} = \boldsymbol{W}\boldsymbol{\beta} + \boldsymbol{\delta},\ \boldsymbol{\delta} \sim (\boldsymbol{0},\ \sigma^2\boldsymbol{I}) \text{ because}$$

$$\begin{aligned} \text{Var}(\boldsymbol{\delta}) &= \text{Var}(\boldsymbol{V}^{-1/2}\boldsymbol{\epsilon}) = \boldsymbol{V}^{-1/2}\sigma^2\boldsymbol{V}\boldsymbol{V}^{-1/2} \\ &= \sigma^2\boldsymbol{V}^{-1/2}\boldsymbol{V}^{1/2}\boldsymbol{V}^{1/2}\boldsymbol{V}^{-1/2} = \sigma^2\boldsymbol{I}. \end{aligned}$$

- Thus, after transformation, we are back to the Gauss-Markov model we are familiar with.
- We can apply all the results we have established previously to the Gauss-Markov model

$$\boldsymbol{z} = \boldsymbol{W}\boldsymbol{\beta} + \boldsymbol{\delta},\ \boldsymbol{\delta} \sim (\mathbf{0}, \sigma^2\boldsymbol{I}).$$

Estimation of $\mathrm{E}(\boldsymbol{y})$ under the Aitken Model

- Note that

$$\mathrm{E}(\boldsymbol{y}) = \mathrm{E}(\boldsymbol{V}^{1/2}\boldsymbol{V}^{-1/2}\boldsymbol{y}) = \boldsymbol{V}^{1/2}\mathrm{E}(\boldsymbol{V}^{-1/2}\boldsymbol{y}) = \boldsymbol{V}^{1/2}\mathrm{E}(\boldsymbol{z}).$$

- Because the Gauss-Markov model holds for $\boldsymbol{z}$, we already know that the best estimate of $\mathrm{E}(\boldsymbol{z})$ is

$$\begin{aligned}
\hat{\boldsymbol{z}} &= \boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{z} = \boldsymbol{W}(\boldsymbol{W}'\boldsymbol{W})^{-}\boldsymbol{W}'\boldsymbol{z} \\
&= \boldsymbol{V}^{-1/2}\boldsymbol{X}((\boldsymbol{V}^{-1/2}\boldsymbol{X})'\boldsymbol{V}^{-1/2}\boldsymbol{X})^{-}(\boldsymbol{V}^{-1/2}\boldsymbol{X})'\boldsymbol{V}^{1/2}\boldsymbol{y} \\
&= \boldsymbol{V}^{-1/2}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{V}^{-1/2}\boldsymbol{V}^{-1/2}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1/2}\boldsymbol{V}^{-1/2}\boldsymbol{y} \\
&= \boldsymbol{V}^{-1/2}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}.
\end{aligned}$$

- Thus, to estimate $\mathrm{E}(\boldsymbol{y}) = \boldsymbol{V}^{1/2}\mathrm{E}(\boldsymbol{z})$, we should use

$$\boldsymbol{V}^{1/2}\hat{\boldsymbol{z}} = \boldsymbol{V}^{1/2}\boldsymbol{V}^{-1/2}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}.$$

- Likewise, if $\boldsymbol{C\beta}$ is estimable, we know the BLUE is the ordinary least squares (OLS) estimator.

$$\begin{aligned} \boldsymbol{C}(\boldsymbol{W}'\boldsymbol{W})^{-}\boldsymbol{W}'\boldsymbol{z} &= \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{V}^{-1/2}\boldsymbol{V}^{-1/2}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1/2}\boldsymbol{V}^{-1/2}\boldsymbol{y} \\ &= \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}. \end{aligned}$$

- $\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} = \boldsymbol{C}\hat{\boldsymbol{\beta}}_{\boldsymbol{V}}$ is called a *Generalized Least Squares* (GLS) estimator.

- $\hat{\beta}_V = (X'V^{-1}X)^{-}X'V^{-1}y$ is a solution to the Aitken Equations:

$$X'V^{-1}Xb = X'V^{-1}y$$

which follow from the Normal Equations

$$\begin{aligned} W'Wb = W'z \quad &\iff \quad X'V^{-1/2}V^{-1/2}Xb = X'V^{-1/2}V^{-1/2}y \\ &\iff \quad X'V^{-1}Xb = X'V^{-1}y. \end{aligned}$$

- Recall that solving the Normal Equations is equivalent to minimizing

$$(\boldsymbol{z} - \boldsymbol{W}\boldsymbol{b})'(\boldsymbol{z} - \boldsymbol{W}\boldsymbol{b}) \text{ over } \boldsymbol{b} \in I\!R^p.$$

- Note that

$$\begin{aligned}
(\boldsymbol{z} - \boldsymbol{W}\boldsymbol{b})'(\boldsymbol{z} - \boldsymbol{W}\boldsymbol{b}) &= (\boldsymbol{V}^{-1/2}\boldsymbol{y} - \boldsymbol{V}^{-1/2}\boldsymbol{X}\boldsymbol{b})'(\boldsymbol{V}^{-1/2}\boldsymbol{y} - \boldsymbol{V}^{-1/2}\boldsymbol{X}\boldsymbol{b}) \\
&= ||\boldsymbol{V}^{-1/2}\boldsymbol{y} - \boldsymbol{V}^{-1/2}\boldsymbol{X}\boldsymbol{b}||^2 \\
&= ||\boldsymbol{V}^{-1/2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})||^2 \\
&= (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b}).
\end{aligned}$$

- Thus, $\hat{\boldsymbol{\beta}}_{\boldsymbol{V}} = (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}$ is a solution to this generalized least squares problem.
- When $\boldsymbol{V}$ is diagonal, the term "Weighted Least Squares" (WLS) is often used instead of GLS.

- An unbiased estimator of σ^2 is

$$
\begin{aligned}
\frac{\boldsymbol{z}'(\boldsymbol{I}-\boldsymbol{P_W})\boldsymbol{z}}{n-\text{rank}(\boldsymbol{W})} &= \frac{\|(\boldsymbol{I}-\boldsymbol{P_W})\boldsymbol{z}\|^2}{n-\text{rank}(\boldsymbol{W})} \\
&= \frac{\|(\boldsymbol{I}-\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{W})^-\boldsymbol{W}')\boldsymbol{z}\|^2}{n-\text{rank}(\boldsymbol{W})} \\
&= \frac{\left\|(\boldsymbol{I}-\boldsymbol{V}^{-1/2}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{V}^{-1/2})\boldsymbol{V}^{-1/2}\boldsymbol{y}\right\|^2}{n-\text{rank}(\boldsymbol{V}^{-1/2}\boldsymbol{X})}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\left\| \boldsymbol{V}^{-1/2}\boldsymbol{y} - \boldsymbol{V}^{-1/2}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} \right\|^2}{n - \text{rank}(\boldsymbol{X})} \\
&= \frac{\left\| \boldsymbol{V}^{-1/2}(\boldsymbol{y} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}) \right\|^2}{n - r} \\
&= \frac{\left\| \boldsymbol{V}^{-1/2}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\boldsymbol{V}}) \right\|^2}{n - r} \\
&= \frac{(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\boldsymbol{V}})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\boldsymbol{V}})}{n - r} \\
&= \hat{\sigma}^2_{\boldsymbol{V}}.
\end{aligned}
$$

Inference Under the Normal Theory Aitken Model

- The Normal Theory Aiken Model:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},\ \boldsymbol{\epsilon} \sim N(\boldsymbol{0}, \sigma^2\boldsymbol{V}).$$

- Under the Normal Theory Aitken Model, we can back transform to convert known formulas in terms of $\boldsymbol{z}$ and $\boldsymbol{W}$ to formulas in terms of $\boldsymbol{y}$ and $\boldsymbol{X}$ to allow inference about estimable $\boldsymbol{C}\boldsymbol{\beta}$ under the Normal Theory Aitken Model.