Equivalence of the Reduced versus Full Model F test and the F test of $C \beta = d$

<table>
<thead>
<tr>
<th>Storage Temperature</th>
<th>20°C</th>
<th>30°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 months</td>
<td>2, 5</td>
<td>9, 12, 15</td>
</tr>
<tr>
<td>6 months</td>
<td>6, 6, 7, 7</td>
<td>16</td>
</tr>
</tbody>
</table>

```r
# R code

time = factor(rep(c(3, 6), each = 5))
temp = factor(rep(c(20, 30, 20, 30), c(2, 3, 4, 1)))
y = c(2, 5, 9, 12, 15, 6, 6, 7, 7, 16)
d = data.frame(time, temp, y)
```
```
<table>
<thead>
<tr>
<th>time</th>
<th>temp</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>30</td>
</tr>
</tbody>
</table>

```

```r
full = lm(y ~ time + temp + time:temp, data = d)
```
```r
model.matrix(full)

(Intercept)  time6  temp30  time6:temp30
1          1     0      0            0  
2          1     0      0            0  
3          1     0      1            0  
4          1     0      1            0  
5          1     0      1            0  
6          1     1      0            0  
7          1     1      0            0  
8          1     1      0            0  
9          1     1      0            0  
10         1     1      1            1

attr("assign")
[1] 0 1 2 3

attr("contrasts")
attr("contrasts")$time
[1] "contr.treatment"

attr("contrasts")$temp
[1] "contr.treatment"
```
### coef(full)

<table>
<thead>
<tr>
<th>(Intercept)</th>
<th>time6</th>
<th>temp30</th>
<th>time6:temp30</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5</td>
<td>3.0</td>
<td>8.5</td>
<td>1.0</td>
</tr>
</tbody>
</table>

# temp 20       temp 30
# -------------------------
# time 3       mu        mu+temp30
#
# time 6       mu+time6  mu+time6+temp30+time6:temp30
#
test=function(lmout,C,d=0){
  b=coef(lmout)
  V=vcov(lmout)
  dfn=nrow(C)
  dfd=lmout$df
  Cb.d=C%*%b-d
  Fstat=drop(t(Cb.d)%*%solve(C%*%V%*%t(C))%*%Cb.d/dfn)
  pvalue=1-pf(Fstat,dfn,dfd)
  list(Fstat=Fstat,pvalue=pvalue)
}

Coverall=matrix(c(
  0,1,0,0,
  0,0,1,0,
  0,0,0,1,
  0,0,0,1
 ),nrow=3,byrow=T)
test(full,Coverall)

$Fstat
[1] 13.53191

$pvalue
[1] 0.004438826

reduced=lm(y~1,data=d)

model.matrix(reduced)

(Intercept)
1    1
2    1
3    1
4    1
5    1
6    1
7    1
8    1
9    1
10   1
rvsf=function(reduced,full) {
  sser=deviance(reduced)
  ssef=deviance(full)
  dfer=reduced$df
  dfef=full$df
  dfn=dfer-dfef
  Fstat=(sser-ssef)/dfn/
    (ssef/dfef)
  pvalue=1-pf(Fstat,dfn,dfef)
  list(Fstat=Fstat,dfn=dnf,dfd=dfef,pvalue=pvalue)
}

rvsf(reduced,full)
$Fstat
[1] 13.53191

$dfn
[1] 3
$dfd$
[1] 6

$pvalue$
[1] 0.004438826

```r
anova(reduced, full)
```

Analysis of Variance Table

Model 1: y ~ 1
Model 2: y ~ time temp time:temp

<table>
<thead>
<tr>
<th></th>
<th>Res.Df</th>
<th>RSS</th>
<th>Df</th>
<th>Sum of Sq</th>
<th>F</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Residuals</td>
<td>1</td>
<td>182.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model</td>
<td>2</td>
<td>23.5</td>
<td>3</td>
<td>159</td>
<td>13.532</td>
<td>0.004439 **</td>
</tr>
</tbody>
</table>

---

Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
Cinteraction=matrix(c(0,0,0,1),nrow=1,byrow=T)

test(full,Cinteraction)

$Fstat
[1] 0.1225532

$pvalue
[1] 0.7382431

anova(full)
Analysis of Variance Table
Response: y

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>1</td>
<td>0.10</td>
<td>0.100</td>
<td>0.0255</td>
<td>0.878292</td>
</tr>
<tr>
<td>temp</td>
<td>1</td>
<td>158.42</td>
<td>158.420</td>
<td>40.4477</td>
<td>0.000709 ***</td>
</tr>
<tr>
<td>time:temp</td>
<td>1</td>
<td>0.48</td>
<td>0.480</td>
<td>0.1226</td>
<td>0.738243</td>
</tr>
<tr>
<td>Residuals</td>
<td>6</td>
<td>23.50</td>
<td>3.917</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
summary(full)

Call:
  lm(formula = y ~ time temp time:temp, data = d)

Residuals:
   Min      1Q  Median      3Q     Max
-3.000e+00 -5.000e-01 -7.606e-17  5.000e-01  3.000e+00

Coefficients:
                         Estimate Std. Error t value Pr(>|t|)
(Intercept)                3.500      1.399   2.501  0.04646 *
time6                        3.000      1.714   1.750  0.13062
temp30                       8.500      1.807   4.705  0.00331 **
time6:temp30                1.000      2.857   0.350  0.73824    ---

Residual standard error: 1.979 on 6 degrees of freedom
Multiple R-squared: 0.8712,   Adjusted R-squared: 0.8068
F-statistic: 13.53 on 3 and 6 DF,  p-value: 0.004439
additive = lm(y ~ time + temp, data = d)
model.matrix(additive)

(Intercept)  time6  temp30
1           1     0      0
2           1     0      0
3           1     0      1
4           1     0      1
5           1     0      1
6           1     1      0
7           1     1      0
8           1     1      0
9           1     1      0
10          1     1      1

attr(, "assign")
[1] 0 1 2
attr(, "contrasts")
attr(, "contrasts")$time
[1] "contr.treatment"

attr(, "contrasts")$temp
[1] "contr.treatment"
coef(additive)

(Intercept)       time6      temp30
  3.26        3.36        8.90

####################################
#          temp 20   temp 30
#          -------------------------
# time 3   mu        mu+temp30
#
# time 6   mu+time6  mu+time6+temp30
#
####################################

rvsf(additive,full)

$Fstat
[1] 0.1225532

$dfn
[1] 1

$dfd
$ pvalue$
[1] 0.7382431

```r
anova(additive, full)
```

Analysis of Variance Table

<table>
<thead>
<tr>
<th>Res.Df</th>
<th>RSS</th>
<th>Df</th>
<th>Sum of Sq</th>
<th>F</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>2</td>
<td>23.98</td>
<td>0.48</td>
<td>0.1226</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>1</td>
<td>23.50</td>
<td>0.48</td>
<td>0.1226</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td></td>
<td></td>
<td>0.7382</td>
<td></td>
</tr>
</tbody>
</table>
library(MASS)
dropterm(full, test="F")
Single term deletions

Model:
y ~ time temp time:temp

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>RSS</th>
<th>AIC</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;none&gt;</td>
<td>23.50</td>
<td>16.544</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>time:temp</td>
<td>1</td>
<td>0.48</td>
<td>23.98</td>
<td>14.746</td>
<td>0.12255</td>
</tr>
</tbody>
</table>

anova(additive)
Analysis of Variance Table

Response: y

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>1</td>
<td>0.10</td>
<td>0.100</td>
<td>0.0292</td>
</tr>
<tr>
<td>temp</td>
<td>1</td>
<td>158.42</td>
<td>158.420</td>
<td>46.2444</td>
</tr>
<tr>
<td>Residuals</td>
<td>7</td>
<td>23.98</td>
<td>3.426</td>
<td></td>
</tr>
</tbody>
</table>

---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
dropterm(additive)
Single term deletions
Single term deletions

Model:
\( y \sim \text{time} + \text{temp} \)

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum of Sq</th>
<th>RSS</th>
<th>AIC</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;none&gt;</td>
<td></td>
<td>23.98</td>
<td>14.746</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>time</td>
<td>1</td>
<td>23.52</td>
<td>47.50</td>
<td>19.581</td>
<td>6.866</td>
<td>0.0343966</td>
</tr>
<tr>
<td>temp</td>
<td>1</td>
<td>158.42</td>
<td>182.40</td>
<td>33.036</td>
<td>46.244</td>
<td>0.0002531</td>
</tr>
</tbody>
</table>

anova(lm(y~temp+time,data=d))
Analysis of Variance Table

Response: y

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>temp</td>
<td>1</td>
<td>135.00</td>
<td>135.000</td>
<td>39.4078</td>
<td>0.000413 ***</td>
</tr>
<tr>
<td>time</td>
<td>1</td>
<td>23.52</td>
<td>23.520</td>
<td>6.8657</td>
<td>0.034397 *</td>
</tr>
<tr>
<td>Residuals</td>
<td>7</td>
<td>23.98</td>
<td>3.426</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Previously we saw how to test for time main effects and temp main effects in the full model by testing $H_0: C \beta = d$.

It is possible but not as easy to test for these main effects using the reduced versus full model approach.

We will use the test for time main effects as an example.

We need to find a matrix whose column space allows for temp main effects and time-by-temp interaction but no time main effects.

It is natural to try the following model specification.
```r
o = lm(y ~ temp + time:temp, data=d)
model.matrix(o)

(Intercept)  temp30  temp20:time6  temp30:time6
 1          1       0            0            0
 2          1       0            0            0
 3          1       1            0            0
 4          1       1            0            0
 5          1       1            0            0
 6          1       0            1            0
 7          1       0            1            0
 8          1       0            1            0
 9          1       0            1            0
10          1       1            0            1

# Examination of this design matrix shows that the cell means are modeled as
#
# | temp 20 | temp 30 |
#|---------|---------|
#| mu      | mu + temp30 |
#| mu + temp20:time6 | mu + temp30 + temp30:time6 |
```
#It is easy to see that this is just the full model in which each treatment group is allowed to have its own mean. Thus, we can't use this code to obtain our reduced model fit.

#One way to obtain the test for time main effects by comparing a reduced and full model is as follows.

```r
full=lm(y~time+temp+time:temp,data=d)

C=matrix(c( 
  + 0,1,0,.5 
  + ),nrow=1,byrow=T)

B=matrix(c( 
  + 1,0,0,0, 
  + 0,0,1,0, 
  + 0,0,0,1, 
  + 0,1,0,.5 
  + ),nrow=4,byrow=T)
```
# Note that $X \beta = X B^{-1} B \beta$.
#
# Let $W = X B^{-1}$ and $\alpha = B \beta$.
#
# Then $C \beta = 0$ is equivalent to $\alpha_4 = 0$. 

W = model.matrix(full) %*% solve(B)

W0 = W[,1:3]

newfull = lm(y ~ W - 1)

newreduced = lm(y ~ W0 - 1)

anova(newreduced, newfull)

Analysis of Variance Table

Model 1: y ~ W0 - 1
Model 2: y ~ W - 1

<table>
<thead>
<tr>
<th>Res.Df</th>
<th>RSS</th>
<th>Df</th>
<th>Sum of Sq</th>
<th>F</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>47.02</td>
<td></td>
<td></td>
<td>23.52</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>23.50</td>
<td>1</td>
<td>23.52</td>
<td>6.0051</td>
</tr>
</tbody>
</table>
rvsf(newreduced,newfull)

$Fstat
[1] 6.005106

$dfn
[1] 1

$dfd
[1] 6

$pvalue
[1] 0.04975481

test(full,C)

$Fstat
[1] 6.005106

$pvalue
[1] 0.04975481