POWER OF THE F-TEST

- Suppose $\boldsymbol{C}$ is a $q \times p$ matrix such that $\boldsymbol{C\beta}$ is testable.
- We have established that

$$F = \frac{(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})' \left[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}'\right]^{-1} (\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})/q}{\hat{\sigma}^2}$$

$$\sim F_{q,n-r}(\delta^2) \text{ where}$$

$$\delta^2 = \frac{(\boldsymbol{C\beta} - \boldsymbol{d})' \left[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}'\right]^{-1} (\boldsymbol{C\beta} - \boldsymbol{d})}{\sigma^2}.$$

- Let $F_{q,n-r,1-\alpha}$ denote the $1-\alpha$ quantile of the central F distribution with q and $n-r$ d.f.

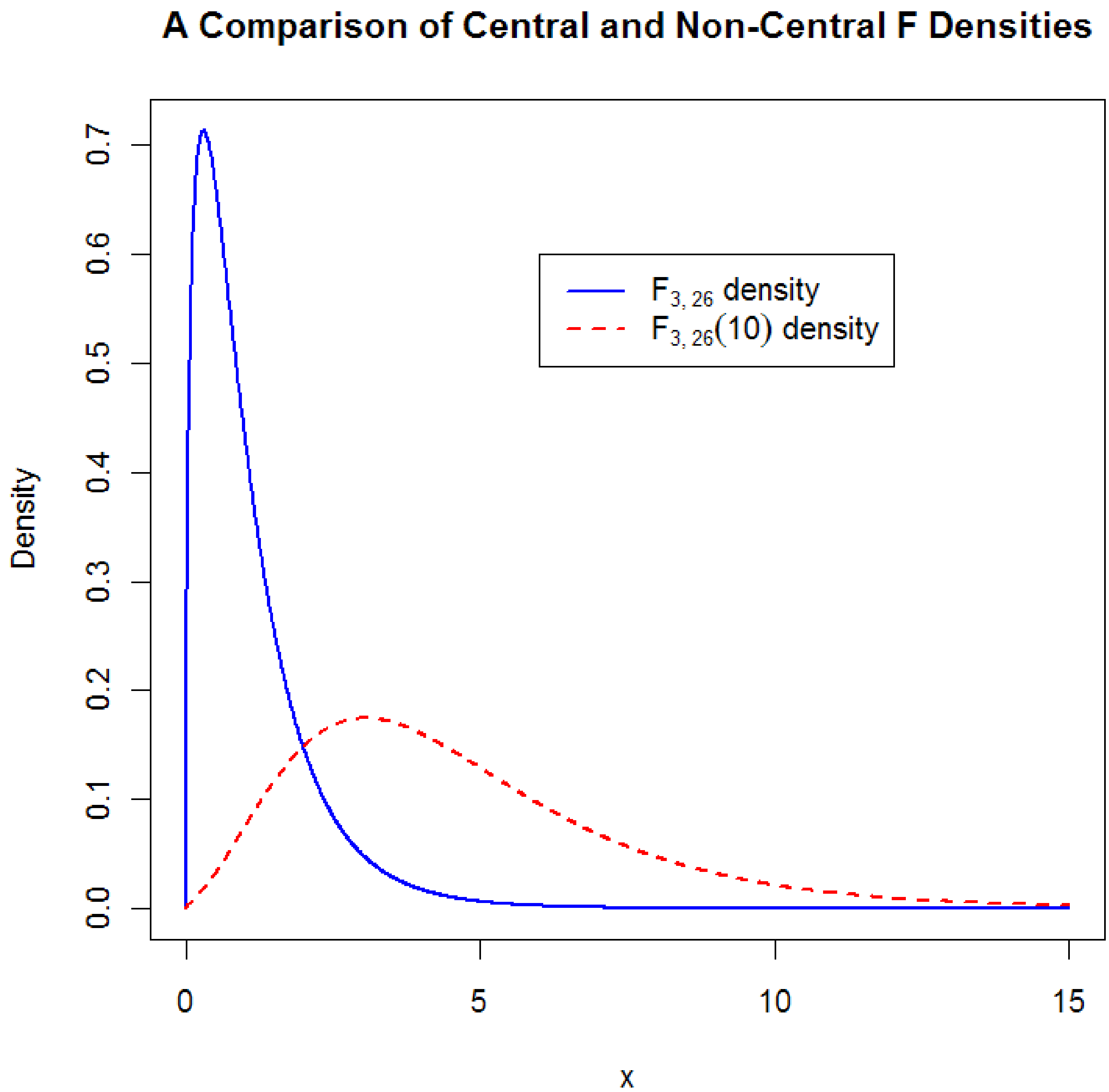
- The power of the significance level α test of

$$H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$$

  is given by

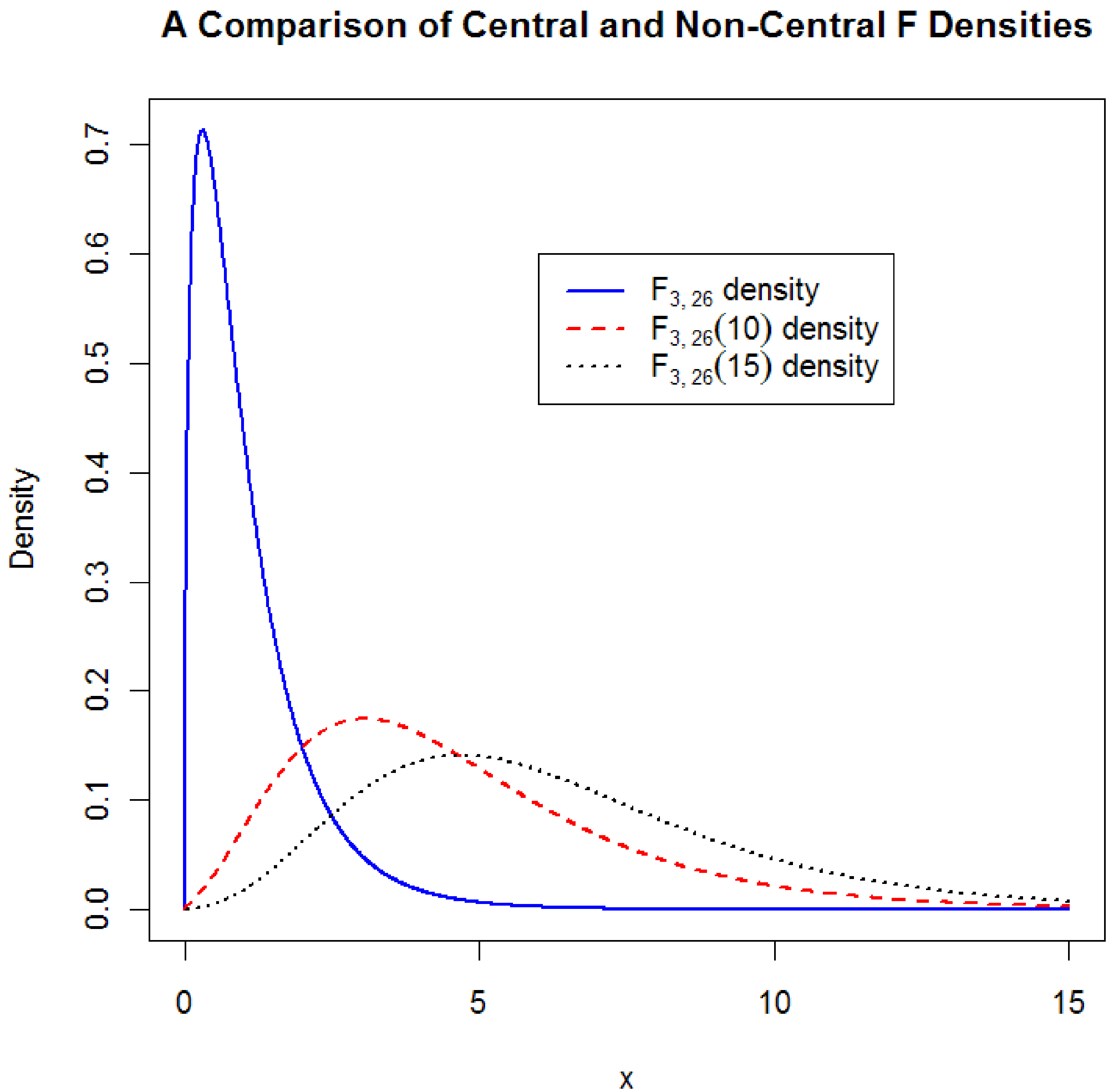
$$P(F \geq F_{q,n-r,1-\alpha}).$$

- The power is an increasing function of the noncentrality parameter δ^2.


A Comparison of Central and Non-Central F Densities
Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
5
10
15
x
$F_{3, 26}$ density
$F_{3, 26}(10)$ density

```
x=seq(0,15,by=.01)
y1=df(x,3,26)
y2=df(x,3,26,ncp=10)

plot(c(x,x),c(y1,y2),pch=" ",
   xlab="x",ylab="Density",
   main="A Comparison of Central and Non-Central
         F Densities")

lines(x,y1,col="blue",lwd=2)
lines(x,y2,lty=2,col="red",lwd=2)

legend(6,.6,
  c(expression(paste(F[list(3,26)]," density")),
    expression(paste(F[list(3,26)](10)," density"))),
    lty=1:2, col=c("blue","red"), lwd=2)
```

```
qf(.95,3,26)

2.975154

1-pf(qf(.95,3,26),3,26,10)

0.6895487
```


A Comparison of Central and Non-Central F Densities
$F_{3,26}$ density
$F_{3,26}(10)$ density
$F_{3,26}(15)$ density
Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
5
10
15
x

```
plot(c(x,x),c(y1,y2),pch=" ",
   xlab="x",ylab="Density",
   main="A Comparison of Central and Non-Central F
         Densities")

lines(x,y1,col="blue",lwd=2)
lines(x,y2,lty=2, col="red",lwd=2)
lines(x,df(x,3,36,15),lty=3, col="black", lwd=2)

legend(6,.6,
  c(expression(paste(F[list(3,26)]," density")),
    expression(paste(F[list(3,26)](10)," density")),
    expression(paste(F[list(3,26)](15)," density"))),
    lty=1:3, col=c("blue","red","black"), lwd=2)

1-pf(qf(.95,3,26),3,26,15)

0.8675232
```

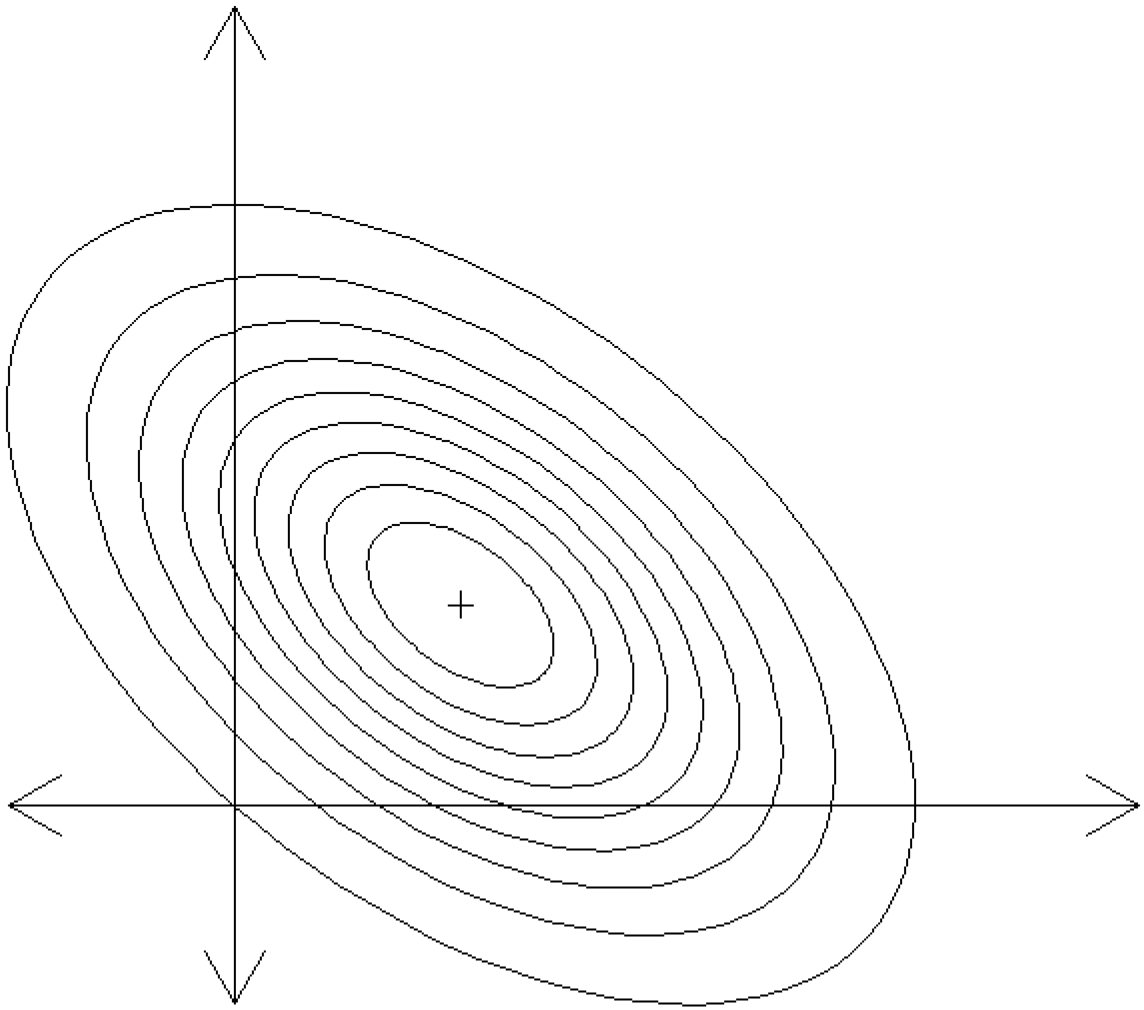
Interpretation of the Noncentrality Parameter

The noncentrality parameter

$$\delta^2 = \frac{(\boldsymbol{C\beta} - \boldsymbol{d})' \left[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{C}'\right]^{-1} (\boldsymbol{C\beta} - \boldsymbol{d})}{\boldsymbol{\sigma}^2}$$

quantifies the discrepancy between $\boldsymbol{C\beta}$ and $\boldsymbol{d}$ with respect to

$$\text{Var}(\boldsymbol{C}\hat{\boldsymbol{\beta}}) = \sigma^2 \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{C}'.$$



All else being equal, the noncentrality parameter

$$\delta^2 = \frac{(\boldsymbol{C\beta} - \boldsymbol{d})' \left[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}'\right]^{-1} (\boldsymbol{C\beta} - \boldsymbol{d})}{\boldsymbol{\sigma}^2}$$

increases as . . .

- the distance between $\boldsymbol{C}\beta$ and $\boldsymbol{d}$ increases,
- σ^2 decreases,
- the design, as defined by $\boldsymbol{X}$, improves (e.g., sample size increases).

- Consider a completely randomized design (CRD) with three treatments.
- Suppose $n = 30$ experimental units are available.
- Let n_i denote the number of experimental units assigned to treatment i for $i =$ 1, 2, 3.
- Suppose we consider the model

$$Y_{ij} = \mu_i + \epsilon_{ij} \quad i = 1, 2, 3; \quad j = 1, \ldots, n_i; \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

- Consider the test of $H_0 : \mu_1 = \mu_2 = \mu_3$.
- This null hypothesis is equivalent to $H_0 : \boldsymbol{C}\boldsymbol{\beta} = \mathbf{0}$, where

$$\boldsymbol{C} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } \boldsymbol{\beta} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}.$$

- (Actually, $\boldsymbol{C}$ could be any 2×3 matrix with row space equal to the row space of the $\boldsymbol{C}$ matrix above.)

$$\boldsymbol{X}'\boldsymbol{X} = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix} \qquad (\boldsymbol{X}'\boldsymbol{X})^{-1} = \begin{bmatrix} \frac{1}{n_1} & 0 & 0 \\ 0 & \frac{1}{n_2} & 0 \\ 0 & 0 & \frac{1}{n_3} \end{bmatrix}$$

$$\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{C}' = \begin{bmatrix} \frac{1}{n_1} + \frac{1}{n_2} & -\frac{1}{n_2} \\ -\frac{1}{n_2} & \frac{1}{n_2} + \frac{1}{n_3} \end{bmatrix}$$

$$\left[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{C}'\right]^{-1} = \begin{bmatrix} \frac{1}{n_2} + \frac{1}{n_3} & \frac{1}{n_2} \\ \frac{1}{n_2} & \frac{1}{n_1} + \frac{1}{n_2} \end{bmatrix} \frac{1}{\frac{1}{n_1 n_2} + \frac{1}{n_1 n_3} + \frac{1}{n_2 n_3}}$$

$$= \begin{bmatrix} \frac{1}{n_2} + \frac{1}{n_3} & \frac{1}{n_2} \\ \frac{1}{n_2} & \frac{1}{n_1} + \frac{1}{n_2} \end{bmatrix} \frac{n_1 n_2 n_3}{n}$$

- Suppose $\mu_1 = 3, \mu_2 = 2, \mu_3 = 1$, and $\sigma^2 = 1$
- Then $\boldsymbol{C\beta} - \boldsymbol{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

$$\begin{aligned} \delta^2 &= [1, 1] \begin{bmatrix} \frac{1}{n_2} + \frac{1}{n_3} & \frac{1}{n_2} \\ \frac{1}{n_2} & \frac{1}{n_1} + \frac{1}{n_2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{n_1 n_2 n_3}{n} \\ &= \frac{4 n_1 n_3 + n_1 n_2 + n_2 n_3}{n}. \end{aligned}$$

- Now suppose $\mu_1 = 3, \mu_2 = \mu_3 = 1, \sigma^2 = 1$
- Then $\boldsymbol{C\beta} - \boldsymbol{d} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and

$$\begin{aligned} \delta^2 &= [2, 0] \begin{bmatrix} \frac{1}{n_2} + \frac{1}{n_3} & \frac{1}{n_2} \\ \frac{1}{n_2} & \frac{1}{n_1} + \frac{1}{n_2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \frac{n_1 n_2 n_3}{n} \\ &= \frac{4n_1(n_2 + n_3)}{n}. \end{aligned}$$

μ_1	μ_2	μ_3	n_1	n_2	n_3	δ^2
3	2	1	5	20	5	10.0
3	2	1	10	10	10	20.0
3	2	1	12	6	12	24.0
3	1	1	5	20	5	$16.\bar{6}$
3	1	1	10	10	10	$26.\bar{6}$
3	1	1	12	6	12	28.8