THE DISTRIBUTION OF $C\hat{\beta}$ WHEN $C\beta$ IS ESTIMABLE

The Multivariate Standard Normal Distribution

Suppose $Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$.

Let $\underset{\sim}{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$.

Then $E(\underset{\sim}{Z}) = \underset{\sim}{0}$ and $Var(\underset{\sim}{Z}) = \underset{n\times n}{I}$.

The random vector $\underset{\sim}{z}$ has the multivariate standard normal distribution denoted

$$\underset{\sim}{z} \sim N(\underset{\sim}{0}, \underset{n \times n}{I}).$$

THE MULTIVARIATE NORMAL DISTRIBUTION

Suppose $\underset{\sim}{z} \sim N(\underset{\sim}{0}, \underset{n \times n}{I})$,

A is an $m \times n$ matrix of constants,

and $\underset{\sim}{\mu}$ is a $m \times 1$ vector of constants.

Then $\underset{\sim}{W} = \underset{\sim}{\mu} + A\underset{\sim}{z}$ has a

multivariate normal distribution.

$$E(\underset{\sim}{w}) = E(\underset{\sim}{\mu} + A\underset{\sim}{z}) = \underset{\sim}{\mu} + E(A\underset{\sim}{z})$$

$$= \underset{\sim}{\mu} + A E(\underset{\sim}{z}) = \underset{\sim}{\mu} + A\underset{\sim}{0}$$

$$= \underset{\sim}{\mu}.$$

$$Var(\underset{\sim}{w}) = Var(\underset{\sim}{\mu} + A\underset{\sim}{z}) = Var(A\underset{\sim}{z})$$

$$= A\, Var(\underset{\sim}{z})\, A' = A I A'$$

$$= AA' \equiv \Sigma$$

We write $\underset{\sim}{w} \sim N(\underset{\sim}{\mu}, \Sigma)$.

When Σ is nonsingular, the probability density function of $\underset{\sim}{w}$ is

$$(2\pi)^{-m/2} |\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(\underset{\sim}{w}-\underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{w}-\underset{\sim}{\mu})\right\}.$$

LINEAR TRANSFORMATIONS OF MULTIVARIATE NORMAL RANDOM VECTORS ARE MULTIVARIATE NORMAL RANDOM VECTORS.

For example, suppose $\underset{\sim}{w} \sim N(\underset{\sim}{\mu}, \underset{m \times m}{\Sigma})$

$\underset{\sim}{a}$ is a $k \times 1$ vector of constants,

and B is a $k \times m$ matrix of constants.

Then $\underset{\sim}{a} + B\underset{\sim}{w} \sim N(\underset{\sim}{a} + B\underset{\sim}{\mu},\ B\Sigma B')$.

To see this, note that

$$E(\underset{\sim}{a} + B\underset{\sim}{w}) = \underset{\sim}{a} + E(B\underset{\sim}{w})$$
$$= \underset{\sim}{a} + B\,E(\underset{\sim}{w})$$
$$= \underset{\sim}{a} + B\underset{\sim}{\mu} \quad \text{and}$$

$$Var(\underset{\sim}{a} + B\underset{\sim}{w}) = Var(B\underset{\sim}{w}) = B\,Var(\underset{\sim}{w})B'$$
$$= B\Sigma B'.$$

Furthermore, $\underset{\sim}{W} \overset{d}{=} \underset{\sim}{\mu} + A\underset{\sim}{Z}$, where

$AA' = \Sigma$. Thus,

$$\underset{\sim}{a} + B\underset{\sim}{W} \overset{d}{=} \underset{\sim}{a} + B(\underset{\sim}{\mu} + A\underset{\sim}{Z})$$

$$= \underset{\sim}{a} + B\underset{\sim}{\mu} + BA\underset{\sim}{Z},$$

which is of the form:

(vector of constants) + (matrix of constants) (multivariate standard normal).

It follows that in the normal theory Gauss-Markov model

$$\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}, \quad \underset{\sim}{\varepsilon} \sim N(\underset{\sim}{0}, \sigma^2 I)$$

$$\underset{\sim}{y} \sim N(X\underset{\sim}{\beta}, \sigma^2 I).$$

Likewise, for estimable $C\underset{\sim}{\beta}$,

$$C\hat{\underset{\sim}{\beta}} = C(X'X)^{-}X'\underset{\sim}{y}$$

must also have a multivariate normal distribution.

To derive $E(C\hat{\beta})$ and $Var(C\hat{\beta})$, we use the fact that $C\beta$ is estimable to write

$$C = AX$$

for some matrix A.

Thus, we have

$$
\begin{aligned}
C\hat{\underset{\sim}{\beta}} &= C(X'X)^{-}X'\underset{\sim}{Y} \\
&= AX(X'X)^{-}X'\underset{\sim}{Y} \\
&= AP_X\underset{\sim}{Y}.
\end{aligned}
$$

$$E(C\hat{\underset{\sim}{\beta}}) = E(AP_X \underset{\sim}{Y})$$

$$= AP_X E(\underset{\sim}{Y})$$

$$= AP_X X \underset{\sim}{\beta}$$

$$= AX\underset{\sim}{\beta}$$

$$= C\underset{\sim}{\beta}.$$

$$
\begin{aligned}
Var(C\hat{\underset{\sim}{\beta}}) &= Var(AP_x \underset{\sim}{y}) \\
&= AP_x \, Var(\underset{\sim}{y}) (AP_x)' \\
&= AP_x (\sigma^2 I) P_x' A' \\
&= \sigma^2 A P_x P_x A' \\
&= \sigma^2 A P_x A' \\
&= \sigma^2 A X (X'X)^{-} X' A' \\
&= \sigma^2 C (X'X)^{-} C'.
\end{aligned}
$$

Thus,

$$C\hat{\underset{\sim}{\beta}} \sim N\left(C\underset{\sim}{\beta},\ \sigma^2 C(X'X)^{-}C'\right).$$