Proof of the Gauss-Markov Theorem
The Gauss-Markov Theorem

Under the Gauss-Markov Linear Model, the OLS estimator $c' \hat{\beta}$ of an estimable linear function $c' \beta$ is the unique Best Linear Unbiased Estimator (BLUE) in the sense that $\text{Var}(c' \hat{\beta})$ is strictly less than the variance of any other linear unbiased estimator of $c' \beta$. 
Unbiased Linear Estimators of $c'\beta$

- If $a$ is a fixed vector, then $a'y$ is a *linear function of* $y$.

- An estimator that is a linear function of $y$ is said to be a *linear estimator*.

- A linear estimator $a'y$ is an unbiased estimator of $c'\beta$ if and only if

\[ E(a'y) = c'\beta \; \forall \; \beta \in \mathbb{R}^p \iff a'E(y) = c'\beta \; \forall \; \beta \in \mathbb{R}^p \]
\[ \iff a'X\beta = c'\beta \; \forall \; \beta \in \mathbb{R}^p \]
\[ \iff a'X = c'. \]
The OLS Estimator of $c'\beta$ is a Linear Estimator

- We have previously defined the Ordinary Least Squares (OLS) estimator of an estimable $c'\beta$ by $c'\hat{\beta}$, where $\hat{\beta}$ is any solution to the normal equations $X'Xb = X'y$.

- We have previously shown that $c'\hat{\beta}$ is the same for any $\hat{\beta}$ that is a solution to the normal equations.

- We have previously shown that $(X'X)^{-}X'y$ is a solution to the normal equations for any generalized inverse of $X'X$ denoted by $(X'X)^{-}$.

- Thus, $c'\hat{\beta} = c'(X'X)^{-}X'y = \ell'y$ (where $\ell' = c'(X'X)^{-}X'$) so that $c'\hat{\beta}$ is a linear estimator.
$c' \hat{\beta}$ is an Unbiased Estimator of an Estimable $c'\beta$

- By definition, $c'\beta$ is estimable if and only if there exists a linear unbiased estimator of $c'\beta$.

- It follows from slide 3 that $c'\beta$ is estimable if and only if $c' = a'X$ for some vector $a$.

- If $c'\beta$ is estimable, then

$$\ell'X = c'(X'X)^{-}X'X = a'X(X'X)^{-}X'X = a'P_XX = a'X = c'.$$

- Thus, by slide 3, $c' \hat{\beta} = \ell'y$ is an unbiased estimator of $c'\beta$ whenever $c'\beta$ is estimable.
Proof of the Gauss-Markov Theorem

Suppose $d'y$ is any linear unbiased estimator other than the OLS estimator $c'\hat{\beta} = \ell'y$.

Then we know the following:

1. $d \neq \ell \iff ||d - \ell||^2 = (d - \ell)'(d - \ell) > 0$, and

2. $d'X = \ell'X = c' \implies d'X - \ell'X = 0' \implies (d - \ell)'X = 0'$.

We need to show $\text{Var}(d'y) > \text{Var}(c'\hat{\beta})$. 
Proof of the Gauss-Markov Theorem

\[
\text{Var}(d'y) = \text{Var}(d'y - c'\hat{\beta} + c'\hat{\beta}) \\
= \text{Var}(d'y - c'\hat{\beta}) + \text{Var}(c'\hat{\beta}) + 2\text{Cov}(d'y - c'\hat{\beta}, c'\hat{\beta}).
\]

\[
\text{Var}(d'y - c'\hat{\beta}) = \text{Var}(d'y - \ell'y) = \text{Var}((d' - \ell')y) = \text{Var}((d - \ell)'y) \\
= (d - \ell)'\text{Var}(y)(d - \ell) = (d - \ell)'(\sigma^2 I)(d - \ell) \\
= \sigma^2(d - \ell)'I(d - \ell) = \sigma^2(d - \ell)'(d - \ell) > 0 \text{ by (1)}.
\]

\[
\text{Cov}(d'y - c'\hat{\beta}, c'\hat{\beta}) = \text{Cov}(d'y - \ell'y, \ell'y) = \text{Cov}((d - \ell)'y, \ell'y) \\
= (d - \ell)'\text{Var}(y)\ell = \sigma^2(d - \ell)'\ell \\
= \sigma^2(d - \ell)'X[(X'X)^{-1}]'c = 0 \text{ by (2)}.
\]
Proof of the Gauss-Markov Theorem

It follows that

\[ \text{Var}(d'y) = \text{Var}(d'y - c'\hat{\beta}) + \text{Var}(c'\hat{\beta}) \]
\[ > \text{Var}(c'\hat{\beta}). \quad \square \]