Question: Why do we look up 0.975 in the t-table when constructing a 95% confidence interval? The 0.975 doesn't seem to match with the 95%. What is going on?

Answer: We have learned that, under fairly general conditions,

$$t = \frac{\bar{Y} - \mu}{s/\sqrt{n}}$$

has a t-distribution with degrees of freedom $n - 1$.

Thus we can find the probability that $t = \frac{\bar{Y}-\mu}{s/\sqrt{n}}$ is between $-t_{n-1}^{(0.975)}$ and $t_{n-1}^{(0.975)}$ by consulting row $n - 1$ of our t-table.

After drawing yourself a picture and doing a little work, you should be able to see that there is a 95% chance that

$$-t_{n-1}^{(0.975)} \leq \frac{\bar{Y} - \mu}{s/\sqrt{n}} \leq t_{n-1}^{(0.975)}.$$

The key is that the area between $-t_{n-1}^{(0.975)}$ and $t_{n-1}^{(0.975)}$ is the middle 95% right around $t = 0$. The area to the right of $t_{n-1}^{(0.975)}$ is 2.5%. Likewise the area to the left of $-t_{n-1}^{(0.975)}$ is 2.5%.

The next several steps involve some basic algebra that you should have learned in high school. You won't be tested on this. By multiplying the left, middle, and right of the equation above by $s/\sqrt{n}$ we can see that there is a 95% chance that

$$-t_{n-1}^{(0.975)} s/\sqrt{n} \leq \bar{Y} - \mu \leq t_{n-1}^{(0.975)} s/\sqrt{n}.$$

By subtracting $\bar{Y}$ from the left, middle, and right of the equation above, we can see that there is a 95% chance that

$$-\bar{Y} - t_{n-1}^{(0.975)} s/\sqrt{n} \leq -\mu \leq -\bar{Y} + t_{n-1}^{(0.975)} s/\sqrt{n}.$$

By multiplying by -1, we can see that there is a 95% chance that

$$\bar{Y} + t_{n-1}^{(0.975)} s/\sqrt{n} \geq \mu \geq \bar{Y} - t_{n-1}^{(0.975)} s/\sqrt{n}.$$

By reordering, we can see that there is a 95% chance that

$$\bar{Y} - t_{n-1}^{(0.975)} s/\sqrt{n} \leq \mu \leq \bar{Y} + t_{n-1}^{(0.975)} s/\sqrt{n}.$$

Thus working with $t_{n-1}^{(0.975)}$ gives us a 95% confidence interval. 95% is our chance of drawing a sample that will lead to a confidence interval that contains μ.