Calculus III
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Textbook: Calculus ninth edition, by Varberg, Purcell, and Rigdon

CHAPTER 14, SUMMARY-FORMULAS

1) A **vector field** is a function that assigns each point x a vector function $F(p)$, where $p = (x, y)$ or $p = (x, y, z)$. Examples of such fields is the electric field, the magnetic field and the gravitational field.

2) The gradient of a scalar field $f(x, y, z)$ defines a vector field which is called **gradient field.** If $\nabla f(x, y, z) = F(x, y, z)$, then f is called a **potential function** and F is called a **conservative vector field**.

3) Let $F = Mi+Nj+Pk$ be a vector field and suppose that $\frac{\vartheta M}{\vartheta x}, \frac{\vartheta N}{\vartheta y}, \frac{\vartheta P}{\vartheta z}$ exist. Then we define

(i) $divF = \nabla \cdot F = \frac{\vartheta M}{\vartheta x} + \frac{\vartheta N}{\vartheta y} + \frac{\vartheta P}{\vartheta z}$

(ii) $curlF = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\vartheta}{\vartheta x} & \frac{\vartheta}{\vartheta y} & \frac{\vartheta}{\vartheta z} \\ M & N & P \end{vmatrix}$

4) A **line integral** is a generalization of a definite integral $\int_a^b f(x)dx$. It is defined over a curve C on the xy-plane (or in space). and we denote by $\int_C f(x, y)ds$. A line integral is a Riemann sum, like definite integrals.

5) If the curve is parametrized by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, then we use the following formula to compute a line integral:
$\boxed{\int_C f(x, y) = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}dt}$ and in 3D
$\boxed{\int_C f(x, y, z) = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt}$.

6) The work done by a force field $F = Mi + Nj + Pk$ in moving a particle along a smooth curve C is given by

$W = \int_C F \cdot dr = \int_C Mdx + Ndy + Pdz$

In this notation, $dr = \langle dx, dy, dz \rangle$.

7) Fundamental Theorem for Line Integrals: Suppose that $r(t)$, $a \leq t \leq b$ parameterizes a smooth curve C. If f is continuously differentiable on an open set containing C, then
$\boxed{\int_C \nabla f \cdot dr = f(b) - f(a)}$, where a, b are the initial point-endpoint of the curve respectively.

8) A set D is called **connected** if any two points in D can be joined by a smooth curve lying entirely in D.

9) Let D be a connected set. The line integral $\int_C F(r) \cdot dr$ is **independent of path** in D if for any points A, B in D it has the same value for every path C in D from A to B.

10)Independence of Path Theorem: Suppose that $F(r)$ is a continuous vector field on an open connected set D. Then, the line integral $\int_C F(r) \cdot dr$ is independent of path if and only if it is conservative, i.e. if and only if there is a scalar field f such that $F(r) = \nabla f(r)$.

11) Theorem: Let $F = Mi + Nj + Pk$ be a vector field on a simply connected set D and assume that M, N, P have continuous first order partial derivatives. Then F is conservative if and only if $curl F = 0$.

If the vector field is two dimensional,i.e. $F = Mi + Nj$ then it conservative if and only if $M_y = N_x$.

12) Green's Theorem: Let C be a smooth, simple, closed curve on the xy-plane, and suppose that C is the boundary of a region S. Let $F = Mi + Nj$ be a vector field and denote by $\oint_C Mdx + Ndy$ the line integral of F over C. Then

$\boxed{\oint_C Mdx + Ndy = \int\int_S \frac{\vartheta N}{\vartheta x} - \frac{\vartheta M}{\vartheta y} dA}$, provided that the partial derivatives exist and are continuous.

13) Vector Forms of Green's Theorem:
If T =unit tangent vector and n =normal vector outward of S, then
(i) $\boxed{\oint_C F \cdot n ds = \int\int_S div F dA}$.

If the vector field denotes the velocity of a fluid, then the above expression gives the flux of F along C, i.e. the amount of fluid leaving S per unit of time.
(ii) $\boxed{\oint_C F \cdot T ds = \int\int_S N_x - M_y dA = \int\int_S (curl F) \cdot k dA}$.

14) Theorem: Let G be a surface given by $z = f(x, y)$, where $(x, y) \in R$. If f has continuous partial derivatives and $g(x, y, z) = g(x, y, f(x, y))$ is continuous on R we compute the surface integral of g by the formula

$$\boxed{\int\int_G g(x, y, z) dS = \int\int_R g(x, y, f(x, y)) \sqrt{f_x^2 + f_y^2 + 1} dy dx}$$

15) Let G be a smooth, 2-sided surface given by $z = f(x, y)$, where $(x, y) \in R$ and n = unit normal upward on G. If f has continuous partial derivatives and $F = Mi + nj + Pk$ is a continuous vector field, then the flux of F across G is given by

$$\boxed{\text{flux}F = \int\int_G F \cdot n dS = \int\int_R [-M f_x - N f_y + P] dx dy}$$

16) Gauss's Divergence Theorem: Let $F = Mi + Nj + Pk$ be a vector field and assume that M, N, P have continuous first-order partial derivatives on a solid S with boundary ϑS. If **n** is the unit normal vector outward to ϑS, then

$$\boxed{\int\int_{\vartheta S} F \cdot n dS = \int\int_S\int div F dV}$$

17) If , β, γ are the direction angles of the normal vector **n**, then Gauss's formula becomes:

$$\boxed{\int\int_S (M\cos\alpha + N\cos\beta + P\cos\gamma)dS = \int\int_S\int (\frac{\vartheta M}{\vartheta x} + \frac{\vartheta N}{\vartheta y} + \frac{\vartheta P}{\vartheta z})dV}$$

18) Stokes's Theorem: Suppose that S is a two-sided surface with a continuously varying normal vector **n**. Suppose also, that the boundary of S, ϑS is a smooth, simple, closed curve. Let $F = Mi + Nj + Pk$ be a vector field, with M, N, P having continuous first-order partial derivatives on S and on ϑS. If T is the unit tangent vector to ϑS, then

$$\boxed{\oint_{\vartheta S} F \cdot T ds = \int\int_S (curl F) \cdot n dS}.$$