Calculus III
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Textbook: Calculus ninth edition, by Varberg, Purcell, and Rigdon

Chapter 13, Summary-formulas

1) If a function $f(x, y)$ is defined on a set S of the $xy-$plane and $f(x, y) \geq 0$ then the double integral $\int\int_S f(x, y)dA$ represents the volume of the solid under the surface $z = f(x, y)$.

2) The double integral of a function $f(x, y)$ on a closed rectangle R is defined by $\int\int_S f(x, y)dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k)\Delta A_k$, (provided that the limit exists)
ΔA_k = area of the k-th subrectangle if we have a partition of R into n subrectangles.

3) Theorem: Every bounded, continuous function on a closed rectangle R is integrable, i.e. the limit of the sum in (2) exists.

4) To evaluate the volume of $f(x, y)$ over a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, we compute the integral $\int_c^d \int_a^b f(x, y)dxdy$. For such rectangles, the order of integration does not matter, i.e.

$$\boxed{\int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx}$$

To integrate with respect to one variable, we think of the other to be constant.

5) A set S is called $y-$**simple** if there are functions $\phi_1(x)$, $\phi_2(x)$ such that $S = \{(x, y) : \phi_1(x) \leq y \leq \phi_2(x), a \leq x \leq b\}$.

A set S is called $x-$**simple** if there are functions $\psi_1(y)$, $\psi_2(y)$ such that $S = \{(x, y) : \psi_1(y) \leq x \leq \psi_2(y), c \leq y \leq d\}$.

6) If we want to change the order of integration over an $y-$simple set S, we need to make it be $x-$ simple, and the opposite. This means

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dxdy.$$

7) Suppose that we want to find the volume under the surface $z = f(x,y)$, $f \geq 0$ over a polar rectangle R of the form $R = \{(r,\theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\}$. Then we can use polar coordinates and calculate as follows:

$$\boxed{\int\int_R f(x,y)dA = \int\int_R f(r\cos\theta, r\sin\theta)rdrd\theta}$$

8) Similarly with (5), a set S is called $r-$**simple** if it has the form $S = \{(r,\theta) : \phi_1(\theta) \leq r \leq \phi_2(\theta), \alpha \leq \theta \leq \beta\}$.
A set S is called $\theta-$**simple** if it has the form $S = \{(r,\theta) : \psi_1(r) \leq \theta \leq \psi_2(r), a \leq r \leq b\}$.

9) Consider a lamina with density $\delta(x,y)$ over a region S. Then we have the following facts:

(i) The **mass m** of the lamina is given by $m = \int\int_S \delta(x,y)dA$.

(ii) The **center of the mass** with coordinates $(\bar{x},\bar{y})$ is given by

$$\bar{x} = \frac{M_y}{m} = \frac{\int\int_S x\delta(x,y)dA}{\int\int_S \delta(x,y)dA} \text{ and } \bar{y} = \frac{M_x}{m} = \frac{\int\int_S y\delta(x,y)dA}{\int\int_S \delta(x,y)dA}.$$

(iii) The **moments of inertia** of the lamina about the x,y,z axes respectively are

(a) $I_x = \int\int_S y^2\delta(x,y)dA$

(b) $I_y = \int\int_S x^2\delta(x,y)dA$

(c) $I_z = I_x + I_y$.

10) Suppose that G is a surface over a closed, bounded region S in the $xy-$plane, and assume that G is the graph of $z = f(x,y)$, where f has continuous first partial derivatives f_x, f_y. Then the surface area of G is given by $\boxed{A(G) = \int\int_S \sqrt{f_x^2 + f_y^2 + 1}dA}$

11) A triple integral is defined for functions of 3 variables. The graphs of these functions are hypersurfaces and we cannot graph them (more dimensions are needed). For a function $f(x, y, z)$, the triple integral over a box B is defined by

$$\int\int_B\int f(x,y,z)dV = \lim_{|P|\to 0}\sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k, \bar{z}_k)\Delta V_k,$$

where ΔV_k is the volume of the k-th subbox, if we consider the partition of B into n subboxes. For triple integrals, there are 6 possible orders of integration. If S is a $3D-$ set, z-simple and the projection of S onto the $xy-$plane is y-simple then

$$\int\int_S\int f(x,y,z)dV = \int_{a_1}^{a_2}\int_{\phi_1(x)}^{\phi_2(x)}\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)dzdydx.$$

12) The triple integral of $f(x, y, z)$ in **cylindrical coordinates** is given by

$$\int\int_S\int f(x,y,z)dV = \int_{\theta_1}^{\theta_2}\int_{r_1(r)}^{r_2(r)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)rdzdrd\theta.$$

13) The triple integral of $f(x, y, z)$ in **spherical coordinates** is given by

$$\int\int_S\int f(x,y,z)dV = \int\int_{somelimits}\int f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi.$$

14) Change of variables in Multiple integrals:

(I) If we have 2 variables x, y and we want to change to the variables u, v and $x = m(u, v)$, $y = n(u, v)$ we define the **Jacobian** of the transformation by

$$J(u,v) = \begin{vmatrix} \frac{\vartheta x}{\vartheta u} & \frac{\vartheta x}{\vartheta v} \\ \frac{\vartheta y}{\vartheta u} & \frac{\vartheta y}{\vartheta v} \end{vmatrix}.$$

Then the transformation is

$$\boxed{\int\int f(x,y)dxdy = \int\int f[m(u,v), n(u,v)]|J(u,v)|dudv}$$

(II) If we have 3 variables x, y, z and we want to change to the variables u, v, w and $x = m(u, v, w)$, $y = n(u, v, w)$,$z = p(u, v, w)$ we define the **Jacobian** of the transformation by

$$J(u, v, w) = \begin{vmatrix} \frac{\vartheta x}{\vartheta u} & \frac{\vartheta x}{\vartheta v} & \frac{\vartheta x}{\vartheta w} \\ \frac{\vartheta y}{\vartheta u} & \frac{\vartheta y}{\vartheta v} & \frac{\vartheta y}{\vartheta w} \\ \frac{\vartheta z}{\vartheta u} & \frac{\vartheta z}{\vartheta v} & \frac{\vartheta z}{\vartheta w} \end{vmatrix}.$$ Then the transformation is

$$\boxed{\int\int\int f(x, y, z)dxdydz = \int\int\int f[m(u, v, w), n(u, v, w), p(u, v, w)]|J(u, v, w)|dudvdw.}$$