Calculus III
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Textbook: Calculus ninth edition, by Varberg, Purcell, and Rigdon

Chapter 12, Summary-formulas

1) Real- valued function of 2 variables: $f = f(x, y)$, for (x, y) in the domain of the function. If $z = f(x, y)$, then z is called the dependent variable.

2) The **level curve** is a projection of the surface $z = f(x, y)$ onto the xy-plane. A collection of such curves is called a contour map.

3) The **partial derivatives** of a function of 2 variables at a point (x_0, y_0) are defined by the limits
$f_x(x_0, y_0) = \lim_{h\to 0} \dfrac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$, where $h = \Delta x$ and
$f_y(x_0, y_0) = \lim_{h\to 0} \dfrac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$, where $h = \Delta y$.

Notation for partial derivatives: $f_x = \frac{\vartheta f}{\vartheta x}$, $f_y = \frac{\vartheta f}{\vartheta y}$. Similarly, for partial derivatives of higher order (and of mixed type): $f_{xx} = \frac{\vartheta^2 f}{\vartheta x^2}$, $f_{yy} = \frac{\vartheta^2 f}{\vartheta y^2}$, $f_{xy} = \frac{\vartheta^2 f}{\theta y \vartheta x}$, $f_{yx} = \frac{\vartheta^2 f}{\vartheta x \vartheta y}$.
Similarly for functions of more variables. To find the partial derivative w.r.t. one variable, we consider the others to be constants.

4) If a function $f(x, y)$ has a limit at (a, b), all the paths (directions) of approach have the same limit value. There are infinitely many ways to approach a point in the plane. If we can find 2 directions with different limit value, then the limit does not exist.

5) A function $f(x, y)$ is continuous at the point (a, b) of its domain if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$, provided that the limit exists.

6) Theorem: If the function $g(x, y)$ is continuous at (a, b) and the function $f(x)$ is continuous at $g(a, b)$ then the composite function $f(g(x, y))$ is continuous at (a, b).

7) A set is said to be **open** if it has only interior points (no boundary points). A set which, in addition, includes all its boundary points is called **closed**.

8) If f_{xy} and f_{yx} are continuous functions on an open set S , then $f_{xy} = f_{yx}$ at any point in S.

9) A function f is differentiable at a point p if it is locally linear at p. If this is true for any point p of an open set S, we say that f is differentiable in S.

10) The "derivative" of a function $f(x, y)$ is the vector $\nabla f = \langle f_x, f_y \rangle$. It is called the **gradient** of f. Similarly, if $f = f(x, y, z)$, then $\nabla f = \langle f_x, f_y, f_z \rangle$.

11) The equation $z = f(p_0) + \nabla f(p_0) \cdot (p - p_0)$ defines the **tangent plane** of the surface $z = f(x, y)$ at p_0.

12) The gradient is a linear operator, i.e. $\nabla(af + bg) = a\nabla f + b\nabla g$, for any scalars a, b. The product rule still holds, i.e. $\nabla fg = f\nabla g + g\nabla f$. Note that f, g are scalar functions of many variables, the gradients are vectors.

13) Theorem: If f is differentiable at a point p, then it is continuous at p. The opposite does not hold.

14) The **directional derivative** $D_u f(p)$ of f at the point p in the direction u is defined by the limit $D_u f(p) = \lim_{h \to 0} \dfrac{f(p + hu) - f(p)}{h}$, where u is a unit vector.

15) Theorem: If f is differentiable at p, then the directional derivative exists for any unit vector u. Furthermore, $D_u f(p) = u \cdot \nabla f(p)$.

16) Maximum rate of change: A function increases most rapidly at p in the direction of the gradient, with rate $|\nabla f(p)|$. A function decreases most rapidly at p in the opposite direction , with rate $-|\nabla f(p)|$.

17) Theorem: The gradient of f at a point P is perpendicular to the level curve of f that passes through P. This can be seen from the fact that f is constant along any level curve.(thus the rate of change is zero).

18) Chain Rule: *First version (with 1 parameter):* If $z = f(x, y)$ is differentiable at x, y and if $x = x(t), y = y(t)$ are differentiable at t, then $\frac{dz}{dt} = \frac{\vartheta f}{\vartheta x}\frac{dx}{dt} + \frac{\vartheta f}{\vartheta y}\frac{dy}{dt}$. The symbol f can be replaced by z.

Second version (with 2 parameters): If $z = f(x, y)$ is differentiable at x, y and if $x = x(s, t), y = y(s, t)$ are differentiable at t, s, then (i) $\frac{\vartheta z}{\vartheta t} = \frac{\vartheta f}{\vartheta x}\frac{\vartheta x}{\vartheta t} + \frac{\vartheta f}{\vartheta y}\frac{\vartheta y}{\vartheta t}$, and (ii) $\frac{\vartheta z}{\vartheta s} = \frac{\vartheta f}{\vartheta x}\frac{\vartheta x}{\vartheta s} + \frac{\vartheta f}{\vartheta y}\frac{\vartheta y}{\vartheta s}$.

19) Suppose that the expression $F(x, y) = 0$ defines y implicitly as a function of x. Then $\frac{dy}{dx} = -\frac{\theta F/\theta x}{\theta F/\theta y}$.

20) Tangent Planes: For the surface $F(x, y, z) = k$, the equation of the tangent plane at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0)\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$. If the surface is given in the form $z = f(x, y)$, then the equation of the tangent plane at$(x_0, y_0, f(x_0, y_0))$ is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

21) Let $z = f(x, y)$, where f is differentiable function, and think of the differentials dx, dy as variables. The **differential** of the dependent variable, dz, is called the differential of f and is given by $z = df(x, y) = f_x(x, y)dx + f_y(x, y)dy$.

22) Suppose that f is a function defined on a domain S.
(i) We say that f has a **global maximum value** at a point p_0 if $f(p_0) \geq f(p)$ for all $p \in S$.
(ii)We say that f has a **global minimum value** at a point p_0 if $f(p_0) \leq f(p)$ for all $p \in S$.
(iii) We say that f has a **global extreme value** at a point p_0 if $f(p_0)$ is either a global maximum or a a global minimum.

Every continuous function defined on a bounded, closed set S attains both a global maximum and a global minimum.

23) Definition: (i) We call p_0 a **stationary point** if p_0 is an interior point of S where f is differentiable and $\nabla f(p_0) = 0$. At such a point, the tangent plane is horizontal.
(ii) We call p_0 a **singular point** if p_0 is an interior point and f is not differentiable there(e.g. a point where the graph of f has a corner).

24) Definition: The **critical points** of f are of three types: boundary points, stationary points or singular points.

25) Theorem: If $f(p_0)$ is extreme value for f then p_0 is a critical point.(one of the three cases in 24).

26) A stationary point which is not an extreme point of a function or a surface is called **saddle point**.

27) Second Partials Test: Suppose that $f(x, y)$ has continuous second partial derivatives in a neighborhood of (x_0, y_0) and that $\nabla f(x_0, y_0) = 0$. Let $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$. Then,
(i) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, $f(x_0, y_0)$ is a local maximum value.
(ii) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, $f(x_0, y_0)$ is a local minimum value.
(iii) If $D < 0$, $f(x_0, y_0)$ is not an extreme value, i.e. (x_0, y_0) is a saddle point.
(iv) If $D = 0$, the test in inconclusive.

28) Lagrange's Method: To maximize or minimize $f(p)$ subject to the constraint $g(p) = 0$, we solve the system of equations $\nabla f(p) = \lambda \nabla g(p)$ and $g(p) = 0$ for p and λ. Each point p is a critical point for the constrained problem and λ is called a *Lagrange multiplier*.