

Calculus II

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Textbook: Calculus ninth edition, by Varberg, Purcell, and Rigdon

CHAPTER 9, SUMMARY-FORMULAS

1) Sequences

A sequence $\{a_n\} = a_1, a_2, \dots, a_n, \dots$, where n is any positive integer, is an ordered list of real numbers that follow a specific pattern. We say that the sequence converges to a limit L as $n \rightarrow \infty$ if the terms of the sequence are very close to the finite number L . If not, we say that the sequence diverges.

2) Remark: A sequence is not exactly a function (we cannot differentiate for example). It simply consists of some discrete numbers of the real line. But still, to find the limit as $n \rightarrow \infty$, we can think of the corresponding formula of a function $f(x)$ and use L'Hopital's Rule if needed.

3) Squeeze Theorem

If $\{a_n\}$ and $\{b_n\}$ are two sequences that converge to the same limit L , and if $\{c_n\}$ is a sequence such that $a_n \leq c_n \leq b_n$ for many n 's then the limit of the sequence $\{c_n\}$ has to be equal to L too.

4) Monotonic Sequence Theorem: If a sequence is nondecreasing and is bounded above by a number U , then the sequence cannot converge to something more than U . Similarly, if a sequence is non-increasing and is bounded below by a number L , then the sequence cannot converge to something less than L .

5) An infinite series is a sum in the form $\sum_{k=1}^{\infty} a_k$ and the n th partial

sum is defined by $S_n = \sum_{k=1}^n a_k$. If the sequence of partial sums $\{S_n\}$ converges to a limit S , then the whole series also converges to S . If $\{S_n\}$ diverges then the series diverges.

6) An infinite series of the form $\sum_{k=1}^{\infty} ar^{k-1}$, where $a \neq 0$ is called geometric series. Such a series converges only if $|r| < 1$. The limit is $\frac{a}{1-r}$

7) N-th term test for divergence If the series $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$. The contrapositive argument is helpful to prove that a series diverges.

8) Theorem: A series $\sum a_k$ of nonnegative terms converges if and only if its partial sums are bounded above.

9) Theorem: Let $f(x)$ be a positive, nonincreasing, continuous function on $[1, \infty)$ and suppose that $a_k = f(k)$ for all positive integers k .

Then the infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if the improper integral $\int_1^{\infty} f(x)$ converges.

10) Some more tests for convergence of positive series:

(A) Suppose that $0 \leq a_n \leq b_n$ for $n \geq N$.

(i) If $\sum b_n$ converges, so does $\sum a_n$.

(ii) If $\sum a_n$ diverges, so does $\sum b_n$.

(B) Suppose that $a_n \geq 0$, $b_n \geq 0$ and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$. If $0 < L < \infty$, then $\sum a_n$, $\sum b_n$ diverge or converge together. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(C) **Ratio Test:** Let $\sum a_n$ be a series of positive terms and suppose that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$.

(i) If $\rho < 1$ the series converges.

(ii) If $\rho > 1$ or if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \infty$, the series diverges.

(iii) If $\rho = 1$, the test is inconclusive.

11) Alternating Series: The assumption that we used in most of the previous convergence-tests was that the terms of the series are positive. Similar results can be obtained for a series with only negative terms because if $a_k < 0$ then $\sum a_k = -\sum (-a_k)$ and we get a series of positive terms again. For series that the terms alternate sign we have

the following test:

Let $a_1 - a_2 + a_3 - a_4 + \dots$ be an alternating series with $a_n > a_{n+1} > 0$. If $\lim_{n \rightarrow \infty} a_n = 0$ then the series converges and the error made by using the sum S_n of the first n terms to approximate S is not more than a_{n+1} .

12) Absolute Ratio test: Let $\sum u_n$ be a series of nonzero terms and suppose that $\lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} = \rho$.

(i) If $\rho < 1$ the series converges.

(ii) If $\rho > 1$ the series diverges.

(iii) If $\rho = 1$ the test is inconclusive.

13) Power Series in x : It has the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

where a_i are constants. The set on which the power series converge (all x) is called the convergent set.

Theorem: The convergent set for a power series in x is always an interval of one of the following 3 types:

(i) The single point $x = 0$.

(ii) An interval $(-R, R)$ (closed or open)

(iii) The whole real line.

Corresponding to the above cases, we say that the series has radius of convergence 0, R or ∞ respectively.

14) A power series converges absolutely on the interior of its interval of convergence.

15) Power Series in $x - a$: $\sum a_n (x - a)^n = a_0 + a_1 (x - a) + a_2 (x - a)^2 + \dots$. The possible convergent sets are

(i) the single point $x = a$.

(ii) The interval $(a - R, a + R)$.

(iii) The whole real line.

Using partial sums, we can differentiate and integrate a power series.

16) Suppose that $f(x)$ is function that is represented as

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots \quad \text{Then } c_n = \frac{f^{(n)}(a)}{n!}.$$

17) Taylor's Formula with Remainder:

Let f be a function whose $(n+1)$ st derivative $f^{(n+1)}(x)$ exists for each x in an open interval I containing a . Then, for each x in I ,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where the remainder $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, and c is some point between x and a . Similar is Taylor's theorem for interval. The error(remainder) R_n goes to zero.

18) Taylor Approximation to a function:

(i) Order 1: $P_1(x) = f(a) + f'(a)(x-a)$.

(ii) Order n : $P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$.

When $a = 0$, we get the Maclaurin polynomial of order n .