

Calculus III

Dimitris Kontogiannis

Textbook: Calculus eighth edition, by Varberg, Purcell, and Rigdon

CHAPTER 16, SUMMARY-FORMULAS

1) If a function $f(x, y)$ is defined on a set S of the xy -plane and $f(x, y) \geq 0$ then the double integral $\int \int_S f(x, y) dA$ represents the volume of the solid under the surface $z = f(x, y)$.

2) The double integral of a function $f(x, y)$ on a closed rectangle R is defined by $\int \int_S f(x, y) dA = \lim_{|P| \rightarrow 0} \sum_{k=1}^n f(\bar{x}_k, \bar{y}_k) \Delta A_k$, (provided that the limit exists)
 ΔA_k = area of the k -th subrectangle if we have a partition of R into n subrectangles.

3) **Theorem:** Every bounded, continuous function on a closed rectangle R is integrable, i.e. the limit of the sum in (2) exists.

4) To evaluate the volume of $f(x, y)$ over a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, we compute the integral $\int_c^d \int_a^b f(x, y) dx dy$. For such rectangles, the order of integration does not matter, i.e.

$$\int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

To integrate with respect to one variable, we think of the other to be constant.

5) A set S is called **y -simple** if there are functions $\phi_1(x), \phi_2(x)$ such that $S = \{(x, y) : \phi_1(x) \leq y \leq \phi_2(x), a \leq x \leq b\}$.

A set S is called **x -simple** if there are functions $\psi_1(y), \psi_2(y)$ such that $S = \{(x, y) : \psi_1(y) \leq x \leq \psi_2(y), c \leq y \leq d\}$.

6) If we want to change the order of integration over an y -simple set S , we need to make it be x -simple, and the opposite. This means

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy dx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx dy.$$

7) Suppose that we want to find the volume under the surface $z = f(x, y)$, $f \geq 0$ over a polar rectangle R of the form $R = \{(r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\}$. Then we can use polar coordinates and calculate as follows:

$$\int \int_R f(x, y) dA = \int \int_R f(r \cos \theta, r \sin \theta) r dr d\theta$$

8) Similarly with (5), a set S is called **r -simple** if it has the form $S = \{(r, \theta) : \phi_1(\theta) \leq r \leq \phi_2(\theta), \alpha \leq \theta \leq \beta\}$.

A set S is called **θ -simple** if it has the form $S = \{(r, \theta) : \psi_1(r) \leq \theta \leq \psi_2(r), a \leq r \leq b\}$.

9) Consider a lamina with density $\delta(x, y)$ over a region S . Then we have the following facts:

(i) The **mass m** of the lamina is given by $m = \int \int_S \delta(x, y) dA$.

(ii) The **center of the mass** with coordinates (\bar{x}, \bar{y}) is given by

$$\bar{x} = \frac{M_y}{m} = \frac{\int \int_S x \delta(x, y) dA}{\int \int_S \delta(x, y) dA} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{\int \int_S y \delta(x, y) dA}{\int \int_S \delta(x, y) dA}.$$

(iii) The **moments of inertia** of the lamina about the x, y, z axes respectively are

(a) $I_x = \int \int_S y^2 \delta(x, y) dA$

(b) $I_y = \int \int_S x^2 \delta(x, y) dA$

(c) $I_z = I_x + I_y$.

10) Suppose that G is a surface over a closed, bounded region S in the xy -plane, and assume that G is the graph of $z = f(x, y)$, where f has continuous first partial derivatives f_x, f_y . Then the surface area of

G is given by $A(G) = \int \int_S \sqrt{f_x^2 + f_y^2 + 1} dA$

11) A triple integral is defined for functions of 3 variables. The graphs of these functions are hypersurfaces and we cannot graph them (more dimensions are needed). For a function $f(x, y, z)$, the triple integral over a box B is defined by

$$\int \int_B \int f(x, y, z) dV = \lim_{|P| \rightarrow 0} \sum_{k=1}^n f(\bar{x}_k, \bar{y}_k, \bar{z}_k) \Delta V_k,$$

where ΔV_k is the volume of the k -th subbox, if we consider the partition of B into n subboxes. For triple integrals, there are 6 possible orders of integration. If S is a $3D$ - set, z -simple and the projection of S onto the xy -plane is y -simple then

$$\int \int_S \int f(x, y, z) dV = \int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x, y, z) dz dy dx.$$

12) The triple integral of $f(x, y, z)$ in **cylindrical coordinates** is given by

$$\int \int_S \int f(x, y, z) dV = \int_{\theta_1}^{\theta_2} \int_{r_1(r)}^{r_2(r)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta.$$

13) The triple integral of $f(x, y, z)$ in **spherical coordinates** is given by

$$\int \int_S \int f(x, y, z) dV = \int \int_{\text{somelimits}} \int f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi.$$

14) Change of variables in Multiple integrals:

(I) If we have 2 variables x, y and we want to change to the variables u, v and $x = m(u, v)$, $y = n(u, v)$ we define the **Jacobian** of the transformation by

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Then the transformation is

$$\int \int f(x, y) dx dy = \int \int f[m(u, v), n(u, v)] |J(u, v)| du dv$$

(II) If we have 3 variables x, y, z and we want to change to the variables u, v, w and $x = m(u, v, w)$, $y = n(u, v, w)$, $z = p(u, v, w)$ we define the **Jacobian** of the transformation by

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}. \text{ Then the transformation is}$$

$$\int \int \int f(x, y, z) dx dy dz = \int \int \int f[m(u, v, w), n(u, v, w), p(u, v, w)] |J(u, v, w)| du dv dw.$$