Calculus III
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Textbook: Calculus eighth edition, by Varberg, Purcell, and Rigdon

Chapter 14, Summary-formulas

1) The distance between 2 points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in three dimensions is given by the formula
$D = |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

2) Standard equation of a sphere with radius r, centered at (h, k, l):

$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.

3) If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are the end points of a line segment, the coordinates of the **midpoint** $M(m_1, m_2, m_3)$ are $m_1 = \frac{x_1+x_2}{2}$, $m_2 = \frac{y_1+y_2}{2}$, $m_3 = \frac{z_1+z_2}{2}$. (i.e. the average of the coordinates of the end points).

4) A **vector** u in three dimensions is described by three numbers u_1, u_2, u_3 which are called the **components** of u. The two notations that we use to write a vector are: $u = \langle u_1, u_2, u_3 \rangle$, or $u = u_1 i + u_2 j + u_3 k$, where $i = \langle 1, 0, 0 \rangle, j = \langle 0, 1, 0 \rangle, k = \langle 0, 0, 1 \rangle$ are the unit vectors (or basis vectors). Same definition as for plane vectors.

5) The **length** (or **magnitude**)$|u|$ of a vector $u = \langle u_1, u_2, u_3 \rangle$ is given by the formula $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$. The length of a vector is always a scalar.

6) The **dot product** of two vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is defined by the formula $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$. Note that the dot product is a scalar, not a vector.
A property that still holds is that $u \cdot u = |u|^2$.

An alternative formula that gives the dot product of u and v is : $u \cdot v = |u||v| \cos\theta$, where θ is the angle between u and v. Formulas are the same with plane vectors.

7) The angles between a vector $a = \langle a_1, a_2, a_3 \rangle$ and the basis vectors i, j, k are called the **direction angles** of a. To find these angles, we use the **direction cosines** $\cos\alpha, \cos\beta, \cos\gamma$ which are given by
$\cos\alpha = \frac{a_1}{|a|}, \cos\beta = \frac{a_2}{|a|}, \cos\gamma = \frac{a_3}{|a|}$. For the direction cosines, the following property holds: $(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1$.

8) The **standard form for the equation of a plane** is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, where (x_1, y_1, z_1) is a point on the plane and $v = \langle A, B, C \rangle$ is a normal vector to the plane (i.e. a perpendicular vector to the plane). We can simplify the above equation by moving the constant terms to the right hand side to get $Ax + By + Cz = D$, for some constant D. Note that the normal vector cannot be the zero vector. This implies that $A^2 + B^2 + C^2 \neq 0$.

9) To find the angle between two planes, we find the angle between the normal vectors of the planes. Note that you can do that by the dot product formula in chapter 13.

10) The distance from a point $P(x_0, y_0, z_0)$ to the plane $Ax + By + Cz = D$ is given by $L = \frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}$

11) The **cross product** of two vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is defined by $u \times v = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$.
The easy way to remember the cross product is in determinant notation:

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

12) The cross product $u \times v$ is perpendicular to both vectors u, v. This means that $u \cdot (u \times v) = v \cdot (u \times v) = 0$

13) An alernative formula for the length of the cross product is $|u \times v| = |u||v| \sin\theta$, where θ is the angle between the 2 vectors. By this formula we see that the cross product is zero if and only if the vectors are parallel. Check theorems B,C in pg.606 for properties.

14) The area of the parallelogram with adjacent sides a, b is given by $A = |a \times b|$ (length of cross product).

15) The volume of a parallelepiped determined by the vectors a, b, c is $V = |a \cdot (b \times c)|$.

16) The **parametric equations of a straight line** in 3 dimensions are $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$, where (x_0, y_0, z_0) is a point on the line and $v = \langle a, b, c \rangle$ is a vector (any vector)parallel to the line. The numbers a, b, c are called the direction numbers of the line.

17) If the directions numbers are nonzero, we can solve the parametric equations for t, to get $t = \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$. These are the **symmetric equations** of the line.

18) If $r(t) = f(t)i + g(t)j + h(t)k$ is the position vector, the direction numbers of the tangent line to the curve are $f'(t), g'(t), h'(t)$. Check the similarities with section 13.4.

19) Similarly with chap.13 if $r(t) = \langle f(t), g(t), h(t)\rangle$ is the position vector for a point P, then we define the following:
(i) velocity $= v(t) = r'(t) = \langle f'(t), g'(t), h'(t)\rangle$,
(ii) acceleration $= a(t) = r''(t) = \langle f''(t), g''(t), h''(t)\rangle$,
(iii) speed $= |v(t)|$.

20) The **curvature** has exactly the same formulas we know: It is given by the expression $k = \frac{|T'(t)|}{|v(t)|} = \frac{|T'(t)|}{|r'(t)|}$, where $T(t) = \frac{r'(t)}{|r'(t)|} = \frac{v(t)}{|v(t)|}$. $T(t)$ is called the **unit tangent vector** of the moving object at a curve.

21) Normal-Tangential components of Acceleration: $a_T = \frac{r' \cdot r''}{|r'|}$, and $a_N = \frac{|r' \times r''|}{|r'|}$. We have also an alternative formula for the curvature k: $k = \frac{|r' \times r''|}{|r'|^3}$.

22) Surface equations:
(i) Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
(ii) Hyperboloid of 1 sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
(iii) Hyperboloid of 2 sheets: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
(iv) Elliptic paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
(v) Hyperbolic paraboloid: $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$
(vi) Elliptic cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

23) Cylindrical- Cartesian coordinates formulas:
(a) Cylindrical to Cartesian: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$
(b)Cartesian to Cylindrical: $r = \sqrt{x^2 + y^2}$, $\tan\theta = \frac{y}{x}$, $z = z$

24) Spherical- Cartesian coordinates formulas:
(a) Spherical to cartesian: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$
(b) Cartesian to Spherical: $\rho = \sqrt{x^2 + y^2 + z^2}$, $\tan\theta = \frac{y}{x}$, $\cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}} = \cos\frac{z}{\rho}$.