ADDENDUM to "Time Optimal Simultaneous Control of Two Level Quantum Systems"

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This note provides the proof of two technical parts of the paper *Time Optimal Simultaneous Control of Two Level Quantum Systems* [1], which have been omitted in that paper for space reason. All the notations are equal to the one in [1].

1 Proof of Lemma 3.5 of [1]

First, we rewrite, for convenience, the equations of the curves $\mathcal{F}_t$ and $\mathcal{S}_t$. We have:

$$x = x_\omega(t) = \cos(\omega\frac{t}{2})\cos(a\frac{t}{2}) + \frac{\omega}{a}\sin(\omega\frac{t}{2})\sin(a\frac{t}{2}), \tag{1}$$

$$y = y_\omega(t) = \sin(\omega\frac{t}{2})\cos(a\frac{t}{2}) - \frac{\omega}{a}\cos(\omega\frac{t}{2})\sin(a\frac{t}{2}), \tag{2}$$

with $a := \sqrt{\omega^2+\gamma^2}$. Then $\mathcal{F}_t$ is the curve $(x_\omega(t), y_\omega(t))$ for $|\omega| \leq \sqrt{\frac{4\pi^2}{t^2}-\gamma^2}$, while $\mathcal{S}_t$ is (1), (2), with $|\omega| > \sqrt{\frac{4\pi^2}{t^2}-\gamma^2}$.

We need to prove the following Lemma:

Lemma: Let $\mathcal{F}_t$ and $\mathcal{S}_t$ the previously defined curves. Then:

1. If $t_1 \neq t_2$ then $\mathcal{F}_{t_1} \cap \mathcal{F}_{t_2} = \emptyset$
2. The curve $\mathcal{F}_t$ does not have self intersections.
3. For any $0 < t < \frac{2\pi}{\gamma}$, we have $\mathcal{F}_t \cap \mathcal{S}_t = \emptyset$.

1.1 If $t_1 \neq t_2$ then $\mathcal{F}_{t_1} \cap \mathcal{F}_{t_2} = \emptyset$

For $\omega = -\sqrt{\frac{4\pi^2}{t^2}-\gamma^2}$ we have that:

$$x = -\cos(\sqrt{4\pi^2-\gamma^2t^2}), \quad y = \sin(\sqrt{4\pi^2-\gamma^2t^2}), \tag{3}$$

which is a point on the unit circle. Since

$$x_{-\omega}(t) = x_\omega(t), \quad y_{-\omega}(t) = -y_\omega(t), \tag{4}$$

the curve $\mathcal{F}_t$ connects two symmetric points on the unit circle. Consider now two curves $\mathcal{F}_{t_1}$ and $\mathcal{F}_{t_2}$, and let us assume $t_1 < t_2$. To prove that these two curves do not have intersections, we actually need to check that they do not have intersections only for $\omega \in \left[-\sqrt{\frac{4\pi^2}{t^2}-\gamma^2}, 0\right]$, since they are symmetric with respect to the x axis (see equation (4)). First, notice that at the endpoint, when $\omega = -\sqrt{\frac{4\pi^2}{t^2}-\gamma^2}$, on the unit circle, equation (3), implies

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that the x coordinate is *strictly decreasing with* t, therefore the intersection point cannot happen on the unit circle. For $\omega = 0$, $y = 0$ for all t, and $x = \cos(\gamma\frac{t}{2})$, which is also strictly decreasing with t. So, if an intersection point between $\mathcal{F}_{t_1}$ and $\mathcal{F}_{t_2}$ occurs it cannot be on the boundary of the unit disc or on the x axis. It has to be for a parameter $\omega := \omega_1$ for $\mathcal{F}_{t_1}$ in $\left(-\sqrt{\frac{4\pi^2}{t_1^2} - \gamma^2}, 0\right)$ and for a parameter $\omega := \omega_2$ for $\mathcal{F}_{t_2}$ in $\left(-\sqrt{\frac{4\pi^2}{t_2^2} - \gamma^2}, 0\right)$. To show that this is also not possible, we first show that it cannot exist a point where $\mathcal{F}_{t_1}$ and $\mathcal{F}_{t_2}$ are tangent to each other. Calculation of $\frac{dx}{d\omega}$ gives

$$\frac{dx}{d\omega} = \frac{\gamma^2}{a^3}\sin(\omega\frac{t}{2})\left[-a\frac{t}{2}\cos(a\frac{t}{2}) + \sin(a\frac{t}{2})\right]; \tag{5}$$

Calculation of $\frac{dy}{d\omega}$ gives

$$\frac{dy}{d\omega} = -\frac{\gamma^2}{a^3}\cos(\omega\frac{t}{2})\left[-a\frac{t}{2}\cos(a\frac{t}{2}) + \sin(a\frac{t}{2})\right]. \tag{6}$$

The function $f(a\frac{t}{2}) = -a\frac{t}{2}\cos(a\frac{t}{2}) + \sin(a\frac{t}{2})$ is always positive for $a\frac{t}{2} \in [\gamma\frac{t}{2}, \pi]$ (recall we are assuming $0 < \gamma\frac{t}{2} < \pi$). Since ω is negative and $|\omega\frac{t}{2}| \leq \sqrt{\pi^2 - \gamma^2\frac{t^2}{4}} < \pi$, $\frac{dx}{d\omega} < 0$ and the curve $\mathcal{F}_t$ gives y as a well defined function of x. Its derivative is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\omega}}{\frac{dx}{d\omega}} = -\cot(\omega\frac{t}{2}). \tag{7}$$

This function is always increasing in the considered interval (it goes from $\cot(\sqrt{\pi^2 - \gamma^2\frac{t^2}{4}})$ to $+\infty$). Therefore if the curves $\mathcal{F}_{t_1}$ and $\mathcal{F}_{t_2}$ coincide so that their slope are the same, we must have

$$\omega_1 t_1 = \omega_2 t_2. \tag{8}$$

Moreover the points where they intersect must coincide. Working in polar coordinates, the radius of a point from equations (1) and (2) is given by $r^2 = 1 - \frac{\gamma^2}{a^2}\sin^2(a\frac{t}{2})$. Equalities of the radiuses gives ($a_{1,2} = \sqrt{\omega_{1,2}^2 + \gamma^2}$).

$$\frac{1}{a_1^2}\sin^2(a_1\frac{t_1}{2}) = \frac{1}{a_2^2}\sin^2(a_2\frac{t_2}{2}). \tag{9}$$

The phase of the point is given by:

$$\psi = \omega\frac{t}{2} + \delta - \arctan\left(\frac{\omega}{a}\tan(a\frac{t}{2})\right) + 2k\pi, \tag{10}$$

where $\delta = 0$ or $\delta = \pi$ according to whether the point is in the first or second quadrant.[1] Therefore we must have

$$\omega_1\frac{t_1}{2} + \delta - \arctan\left(\frac{\omega_1}{a_1}\tan(a_1\frac{t_1}{2})\right) + 2k_1\pi = \omega_2\frac{t_2}{2} + \delta - \arctan\left(\frac{\omega_2}{a_2}\tan(a_2\frac{t_2}{2})\right) + 2k_2\pi. \tag{11}$$

Since $|\omega_{1,2}t_{1,2}| \leq \sqrt{\pi^2 - \gamma^2\frac{t_{1,2}^2}{4}} < \pi$ the absolute value of the sum of the first and third terms in both left and right hand side formula (11) is bounded by $\frac{3\pi}{2}$. So the above equality (11) may only occur for $k_1 = k_2$. This implies

$$\omega_1\frac{t_1}{2} - \arctan\left(\frac{\omega_1}{a_1}\tan(a_1\frac{t_1}{2})\right) = \omega_2\frac{t_2}{2} - \arctan\left(\frac{\omega_2}{a_2}\tan(a_2\frac{t_2}{2})\right). \tag{12}$$

Using (8) and the fact that arctan is an increasing function, we obtain,

$$\frac{\omega_1}{a_1}\tan(a_1\frac{t_1}{2}) = \frac{\omega_2}{a_2}\tan(a_2\frac{t_2}{2}). \tag{13}$$

Using (9) and the fact that $0 < a\frac{t}{2} < \pi$, we obtain from (13)

$$\frac{\cos^2(a_1\frac{t_1}{2})}{\omega_1^2} = \frac{\cos^2(a_2\frac{t_2}{2})}{\omega_2^2}. \tag{14}$$

[1] To be more precise, we should have replaced $\arctan\left(\frac{\omega}{a}\tan(a\frac{t}{2})\right)$ with $-\frac{\pi}{2}$ in the case where $a\frac{t}{2} = \frac{\pi}{2}$. The treatment of this special case is analogous to the treatment of the case where $a\frac{t}{2} \neq \frac{\pi}{2}$. So we focus on this case only.

Combining (9) and (14), we obtain

$$1=\frac{\omega_1^2}{\omega_2^2}\cos^2(a_2\frac{t_2}{2})+\frac{a_1^2}{a_2^2}\sin^2(a_2\frac{t_2}{2}). \tag{15}$$

From (8) and $t_1 < t_2$ we obtain $\frac{\omega_1^2}{\omega_2^2}>1$ and $\frac{a_1^2}{a_2^2}>1$. Therefore in (15), we have

$$1>\cos^2(a_2\frac{t_2}{2})+\sin^2(a_2\frac{t_2}{2})=1, \tag{16}$$

which is a contradiction. So, if an intersection point between $\mathcal{F}_{t_1}$ and $\mathcal{F}_{t_2}$ occurs, at the intersection the two curves are not tangent. Now we use this to prove that intersections are not possible. Denote by $C_1=\{t\in(0,t_2)\mid \mathcal{F}_t\cap\mathcal{F}_{t_2}\neq\emptyset\}$, and by C_2 its complement. We need to show that $C_1=\emptyset$. We will prove that this set is both closed and open, so it must be empty since the interval $(0,t_2)$ is a connected set. First we prove that C_1 is open. Denote by $\mathcal{A}$ the region bounded by the unit circle, the $x-$axis, and the curve $\mathcal{F}_{t_2}$. If a curve $\mathcal{F}_{\tilde{t}}$, for $0<\tilde{t}<t_2$ intersects the curve $\mathcal{F}_{t_2}$, since at the intersection point the two curves are not tangent, this implies that the curve $\mathcal{F}_{\tilde{t}}$ leaves the region $\mathcal{A}$. Let $Q=\big(x_{\tilde{\omega}}(\tilde{t}),y_{\tilde{\omega}}(\tilde{t})\big)\in\mathcal{F}_{\tilde{t}}$ and outside $\mathcal{A}$. Let V be a neighborhood of Q which lies outside $\mathcal{A}$. By continuity there exits $\epsilon>0$ such that if $|t-\tilde{t}|\le\epsilon$ and $|\omega-\tilde{\omega}|<\epsilon$, then $(x_\omega(t),y_\omega(t))\in V$. By choosing, if necessary, $\epsilon_1<\epsilon$, such that $\omega\in\left[-\sqrt{\frac{4\pi^2}{t^2}-\gamma^2},0\right]$, for $|t-\tilde{t}|\le\epsilon_1$ and $|\omega-\tilde{\omega}|<\epsilon_1$, we have that for $t\in(\tilde{t}-\epsilon_1,\tilde{t}+\epsilon_1)$ the curve $\mathcal{F}_t$ reaches V, so goes outside $\mathcal{A}$, and so must intersect $\mathcal{F}_{t_2}$. Thus C_1 is open. Now we prove that also C_2 (the complement of C_1) is an open set. If $\tilde{t}\in C_2$, then the curve $\mathcal{F}_{\tilde{t}}$ lies all inside the interior of $\mathcal{A}$. Thus there exists a neighborhood W of this curve which lies all inside $\mathcal{A}$. For all $\tilde{\omega}\in\left[-\sqrt{\frac{4\pi^2}{\tilde{t}^2}-\gamma^2},0\right]$, there exists $\delta>0$ such that if $|t-\tilde{t}|\le\delta$, and $|\omega-\tilde{\omega}|<\delta$ then $(x_\omega(t),y_\omega(t))\in W$. The constant δ depends on $\tilde{t}$ and also on $\tilde{\omega}$. Since $\tilde{\omega}$ varies in a compact set, we may choose a common $\delta>0$, thus all the curves $\mathcal{F}_t$ for $t\in(\tilde{t}-\delta,\tilde{t}+\delta)$ lie in W, so in particular they do not intersect $\mathcal{F}_{t_2}$. So $(\tilde{t}-\delta,\tilde{t}+\delta)\subset C_2$, so C_2 is open, and this implies that C_1 is closed.

1.2 The curve $\mathcal{F}_t$ does not have self intersections.

Consider the function $r^2=1-\frac{\gamma^2}{a^2}\sin^2(a\frac{t}{2})$ seen as a function of ω. This function is even and it is easily seen, by taking the derivative with respect to ω, that it is decreasing in the interval $[-\sqrt{\frac{4\pi^2}{t^2}-\gamma^2},0]$ (so consequently increasing in the interval $[0,\sqrt{\frac{4\pi^2}{t^2}-\gamma^2}]$). Thus a possible self intersection may only be for two opposite values of ω. On the other hand, by equation (4), to have equality we must suppose $y_\omega(t)=0$ and this can happen only for $\omega=0$.[2] So $\mathcal{F}_t$ does not have self intersections.

1.3 For any $0<\frac{t}{2}<\frac{\pi}{\gamma}$, we have $\mathcal{F}_t\cap\mathcal{S}_t=\emptyset$.

We consider positive ω because of the symmetry of the curve (1) (2). We want to show that $\mathcal{S}_t$ remains strictly inside the region of the unit disc bounded by $\mathcal{F}_t$ and the boundary of the unit disc. Denote this region by A. This shows in particular that it has no intersection with $\mathcal{F}_t$. As in the proof of Theorem 1 of the paper, for any given t, $\lim_{\omega\to\infty}x(\omega)=1$ and $\lim_{\omega\to\infty}y(\omega)=0$. Therefore $\mathcal{S}_t$ eventually belongs to this region. Given a point $(x_\omega(t),y_\omega(t))$ of the curve (1) (2), we denote by $\phi(\omega)$ its phase if it belongs to $\mathcal{F}_t$, i.e. $\omega\in\left[0,\sqrt{\frac{4\pi^2}{t^2}-\gamma^2}\right]$, and we

[2] This is easily seen as follows: If $y=0$, from (2) we have $\sin(\omega\frac{t}{2})\cos(a\frac{t}{2})=\frac{\omega}{a}\cos(\omega\frac{t}{2})\sin(a\frac{t}{2})$. From this, if $\cos(\omega\frac{t}{2})=0$ then $\cos(a\frac{t}{2})=0$ and viceversa. Therefore in this case, we would have (take positive ω) $\omega t=k\pi$ and $at=l\pi$ with k and l odd and strictly positive (since we are assuming $\omega\neq0$). Solving for t, we get $\frac{\omega}{a}=\frac{k}{l}$ and using $a=\sqrt{\omega^2+\gamma^2}$ we get $(l^2-k^2)\omega^2=k^2\gamma^2$ which shows that $l>k$. From this, we also obtain $\omega=\sqrt{\frac{k^2}{l^2-k^2}}\gamma$ which replaced in $\omega t=k\pi$ gives $t=\frac{\pi\sqrt{l^2-k^2}}{\gamma}$ that along with the fact that l and k are odd and l is strictly greater than k contradicts the fact that $\frac{t}{2}<\frac{\pi}{\gamma}$. Therefore, we can write $\frac{\tan(\omega\frac{t}{2})}{\omega t}=\frac{\tan(a\frac{t}{2})}{at}$. Since for the given bounds on the value of $\omega\frac{t}{2}$ is $0<\omega\frac{t}{2}<\sqrt{\pi^2-\gamma^2\frac{t^2}{4}}<\pi$, and the value of $a\frac{t}{2}$ is $\gamma\frac{t}{2}<a\frac{t}{2}<\pi$, so the equality $\frac{\tan(\omega\frac{t}{2})}{\omega t}=\frac{\tan(a\frac{t}{2})}{at}$ means that $\omega\frac{t}{2}$ and $a\frac{t}{2}$ are both in $(0,\frac{\pi}{2})$ or both in $(\frac{\pi}{2},\pi)$. On both these intervals the function $\frac{\tan(x)}{x}$ is increasing. Therefore we have $\omega t=at$ which is possible only if $\omega=0$ and $\gamma=0$.

denote by $\psi(\omega)$ its phase if it belongs to $\mathcal{S}_t$, i.e. $\omega \in \left(\sqrt{4\frac{\pi^2}{t^2} - \gamma^2}, +\infty\right)$. Using (1) (2), the phase of this initial point, i.e. when $\omega = 0$, is $\phi(0) = 0$ if $a\frac{t}{2} = \gamma\frac{t}{2} \leq \frac{\pi}{2}$ and $\phi(0) = -\pi$ if $\frac{\pi}{2} < a\frac{t}{2} = \gamma\frac{t}{2} < \pi$. The portion of the curve (1) (2) belonging to $\mathcal{F}_t$, corresponds $a\frac{t}{2}$ going from $\gamma\frac{t}{2}$ to π. To prove our statement, and show that, in fact, $\mathcal{S}_t$ never exits the region A, we will consider separately the two cases:

- $\gamma\frac{t}{2} > \frac{\pi}{2}$, in which case the initial point corresponding to $\omega = 0$ is on the negative side of the x axis, and $\phi(0) = -\pi$,
- $\gamma\frac{t}{2} \leq \frac{\pi}{2}$ in which case it is on the positive axis, and $\phi(0) = 0$. The proof in this second case requires few extra elements.

Case: $\gamma t > \pi$
We will prove that:

a. The value of the phase, $\psi(\omega)$, of the corresponding point in $\mathcal{S}_t$ is always greater than the value of the phase (for $\mathcal{F}_t$) at the point corresponding to $\omega = \sqrt{\frac{4\pi^2}{t^2} - \gamma^2}$ which is (cf.(10)),

$$\phi_0 := -\pi + \sqrt{\pi^2 - \gamma^2\frac{t^2}{4}}. \tag{17}$$

b. This value ϕ_0 is always greater than all values of the phase, $\phi(\omega)$, of the corresponding points in $\mathcal{F}_t$.[3]

This gives that the phase for points on $\mathcal{S}_t$, $\psi(\omega)$, is always greater than the phase for points on $\mathcal{F}_t$, $\phi(\omega)$, and therefore intersection cannot occur.

a. We first consider the function $\psi(\omega)$, where ω varies so that $a\frac{t}{2} \in [\pi, \frac{3}{2}\pi]$ and then, for $k = 2, 3, ...,$ for $a\frac{t}{2} \in \left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right]$. If $a\frac{t}{2} \in [\pi, \frac{3}{2}\pi)$ the phase is

$$\psi(\omega) = \omega\frac{t}{2} - \pi - \arctan\left(\frac{\omega}{a}\tan(a\frac{t}{2})\right). \tag{18}$$

Calculating $\frac{d\psi}{d\omega}$ we obtain, after some manipulations,

$$\frac{d\psi}{d\omega} = \frac{t}{2} - \frac{a^2\cos^2(a\frac{t}{2})}{\gamma^2\cos^2(a\frac{t}{2}) + \omega^2}\left(\frac{\gamma^2}{a^3}\tan(a\frac{t}{2}) + \frac{\omega^2\frac{t}{2}}{a^2\cos^2(a\frac{t}{2})}\right). \tag{19}$$

To study the sign of $\frac{d\psi}{d\omega}$, we calculate $(\gamma^2\cos^2(a\frac{t}{2}) + \omega^2)a\frac{d\psi}{d\omega}$, whose sign is the same as the sign of $\frac{d\psi}{d\omega}$. We have

$$(\gamma^2\cos^2(a\frac{t}{2}) + \omega^2)a\frac{d\psi}{d\omega} = \gamma^2\cos(a\frac{t}{2})(\cos(a\frac{t}{2})a\frac{t}{2} - \sin(a\frac{t}{2})) := f(a\frac{t}{2}). \tag{20}$$

For $a\frac{t}{2} \in (\pi, \frac{3\pi}{2})$ this function is positive and then becomes negative with increasing $a\frac{t}{2}$. It has one zero and $\lim_{a\frac{t}{2} \to \frac{3\pi}{2}} f(a\frac{t}{2}) = 0$. Therefore, to show that $\psi(\omega)$ is greater than the one corresponding to the value of $\omega = \sqrt{\frac{4\pi^2}{t^2} - \gamma^2}$, i.e., the value where $a\frac{t}{2} = \pi$, for all values of $a\frac{t}{2} \in (\pi, \frac{3\pi}{2}]$ is enough to show that the phase for $a\frac{t}{2} = \frac{3\pi}{2}$ is greater than the phase for $a\frac{t}{2} = \pi$. This gives the inequality:

$$\phi_0 = -\pi + \sqrt{\pi^2 - \gamma^2\frac{t^2}{4}} < -\frac{3\pi}{2} + \sqrt{\frac{9\pi^2}{4} - \gamma^2\frac{t^2}{4}}. \tag{21}$$

For future use we prove the more general inequality

$$\phi_0 = -\pi + \sqrt{\pi^2 - \gamma^2\frac{t^2}{4}} < -\frac{(2k+1)\pi}{2} + \sqrt{\frac{(2k+1)^2\pi^2}{4} - \gamma^2\frac{t^2}{4}}, \tag{22}$$

[3] Here we take the 'principal' value of the phase, i.e., $\psi \in [-\pi, \pi)$, and in this case, since we are in the negative y's region $\psi \in [-\pi, 0]$.

$k = 1, 2, ...$ of which (21) is a special case with $k = 1$. Inequality (22) is equivalent to

$$(2k-1)\frac{\pi}{2} + \sqrt{\pi^2 - \gamma^2 \frac{t^2}{4}} < \sqrt{\frac{(2k+1)^2\pi^2}{4} - \gamma^2 \frac{t^2}{4}}. \tag{23}$$

Squaring both terms and after some simplifications, we obtain,

$$(2k-1)^2\pi + 4\pi + 4(2k-1)\sqrt{\pi^2 - \gamma^2 \frac{t^2}{4}} < (2k+1)^2\pi, \tag{24}$$

which, after some manipulations gives, the obviously true relation

$$\sqrt{\pi^2 - \gamma^2 \frac{t^2}{4}} < \pi. \tag{25}$$

Generalizing (18), we define

$$\omega \frac{t}{2} - k\pi - \arctan(\frac{\omega}{a}\tan(a\frac{t}{2})) := \psi_k(\omega), \tag{26}$$

when $a\frac{t}{2} \in \left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right)$, for $k = 1, 2, 3, ...$ The general expression for the phase when $a\frac{t}{2} \in [\frac{3\pi}{2}, \infty)$ is given by the continuous function $\psi = \psi(\omega)$

$$\psi(\omega) = \psi_k(\omega), \tag{27}$$

when $a\frac{t}{2} \in \left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right)$, for $k = 2, 3, ...$,

$$\psi(\omega) = \lim_{\omega \to \sqrt{(2k-1)^2 \frac{\pi^2}{t^2} - \gamma^2}^{+}} \psi_{k-1}(\omega) = \sqrt{(2k-1)^2 \frac{\pi^2}{4} - \gamma^2 \frac{t^2}{4}} - k\pi + \frac{\pi}{2}, \tag{28}$$

in the points $a\frac{t}{2} = \frac{(2k-1)\pi}{2}$, $k = 2, 3, ...$[4] In the interior of the interval where $a\frac{t}{2} \in \left[(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right]$, for $k = 2, 3, ...$, the derivative of the function $\psi(\omega)$ is still given by (19) and it is positive, then, it becomes negative and then it tends again to zero as $a\frac{t}{2} \to \frac{(2k+1)\pi}{2}$. This means that $\psi(\omega)$ is increasing, it reaches a maximum and then it decreases to a minimum. The maximum cannot be greater than zero because by continuity the curve (1) (2) will have to cross the x axis and we have seen that this happens only when $\omega = 0$. This ensures that $\psi(\omega) < 0$. The minimum the function tends to is the value of the function at $a\frac{t}{2} = (2k+1)\frac{\pi}{2}$, i.e., $\sqrt{(2k+1)^2 \frac{\pi^2}{4} - \gamma^2 \frac{t^2}{4}} - (2k+1)\frac{\pi}{2}$, which is greater than $\phi_0 := -\pi + \sqrt{\pi^2 - \gamma^2 \frac{t^2}{4}}$ (and therefore of $-\pi$) as it was proved in (22).

b. The phase $\phi(\omega)$ of a point belonging to $\mathcal{F}_t$, which corresponds to $a\frac{t}{2}$ going from $\gamma\frac{t}{2}$ to π, can be written, since we have $\gamma\frac{t}{2} > \frac{\pi}{2}$:

$$\phi(\omega) = \omega\frac{t}{2} - \pi - \arctan\left(\frac{\omega}{a}\tan(a\frac{t}{2})\right). \tag{29}$$

This expression is the same as the one given in equation (18). Thus if we take the derivative with respect to ω, we end up with the same $f(a\frac{t}{2})$ function of equation (20). This function is positive for $(\frac{\pi}{2} <)\gamma\frac{t}{2} < a\frac{t}{2} < \pi$. Thus the phase is always increasing so $\phi(\omega) \leq \phi_0$.

Case: $\gamma t \leq \pi$

In this case, the previous argument has to be modified with some extra elements since the phase $\phi(\omega)$ of the points of $\mathcal{F}_t$, is not always increasing. Here we have:

$$\phi(\omega) = \omega\frac{t}{2} - \arctan\left(\frac{\omega}{a}\tan(a\frac{t}{2})\right), \tag{30}$$

[4] We remark that this is the 'principal' value of the phase, that is, the one with value in the interval $(-\pi, 0)$. To see this, notice that at the first endpoint of the interval $[(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}]$, we have $\psi(\omega)$ given by $\sqrt{(2k-1)^2\frac{\pi^2}{4} - \gamma^2\frac{t^2}{4}} - k\pi + \frac{\pi}{2}$ as in (28) and we can see $0 > \sqrt{(2k-1)^2\frac{\pi^2}{4} - \gamma^2\frac{t^2}{4}} - k\pi + \frac{\pi}{2} > -\pi$. The right inequality is equivalent to $\sqrt{(2k-1)^2\frac{\pi^2}{4} - \gamma^2\frac{t^2}{4}} > k\pi - \frac{3\pi}{2}$, which squaring both terms and after simplifications leads to the true inequality $-\gamma^2\frac{t^2}{4} > -2(k-1)\pi^2$. The left inequality follows similarly. The fact that the phase $\psi = \psi(\omega)$ remains in this range is a consequence of the considerations that follow.

if $a\frac{t}{2} \in [\gamma\frac{t}{2}, \frac{\pi}{2})$. Then $\phi(\omega)$ is given by equation (29), for $a\frac{t}{2} \in [\frac{\pi}{2}, \pi]$. The function $\phi(\omega)$ has a minimum for $a\frac{t}{2} = \frac{\pi}{2}$ and then increases for $a\frac{t}{2} \in (\frac{\pi}{2}, \pi]$. Moreover, we have that $\phi(\frac{\pi}{2}) = \phi_0$ in (17). The phase $\psi(\omega)$ of the points in $\mathcal{S}_t$, is again given by equations (26), (27), (28), for $a\frac{t}{2} \in (\pi, \infty)$. Proceeding as in the previous case (part a.), we show that the phase for the points of $\mathcal{S}_t$ has to be always greater than the value ϕ_0, and therefore greater than the one of all points on $\mathcal{F}_t$ for $a\frac{t}{2} \in (\frac{\pi}{2}, \pi]$. Therefore if an intersection occurs it has to occur in the first part of the curve $\mathcal{F}_t$, the one corresponding to values of $a\frac{t}{2} \in [\gamma\frac{t}{2}, \frac{\pi}{2})$. Now consider the square radius r^2 as a function of $a\frac{t}{2}$, i.e.,

$$r^2 = 1 - \frac{\gamma^2 \frac{t^2}{4}}{a^2 \frac{t^2}{4}} \sin^2(a\frac{t}{2}). \tag{31}$$

For $a\frac{t}{2} \in [\gamma\frac{t}{2}, \pi]$ this function is increasing as a function of $a\frac{t}{2}$. Therefore the radius on the first part of $\mathcal{F}_t$ is $\leq$ than the value of this function at $a\frac{t}{2} = \frac{\pi}{2}$, which is,

$$r_0^2 = 1 - \frac{4\gamma^2 \frac{t^2}{4}}{\pi^2}. \tag{32}$$

Now for $a\frac{t}{2} \in [\pi, \frac{3\pi}{2}]$ the square radius function is decreasing, it has a minimum and then it is increasing again. A computation of the derivative with respect to $a\frac{t}{2}$ shows that the minimum is at the point $a\frac{t}{2} = z_1 \in (\pi, \frac{3\pi}{2})$ such that $\tan(z_1) = z_1$. At this point, writing $\sin^2(z_1) = \frac{\tan^2(z_1)}{1+\tan^2(z_1)}$, we can write the radius in (31) as

$$r_1^2 = 1 - \frac{\gamma^2 \frac{t^2}{4}}{1 + z_1^2}. \tag{33}$$

Comparing with (32) using the fact that $z_1 > \pi$, we obtain $r_1 > r_0$, i.e., the minimum radius for $a\frac{t}{2} \in [\pi, \frac{3\pi}{2}]$ is greater than the value r_0. This argument extends to vales of $a\frac{t}{2} \in [(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}]$, for any $k \geq 2$. In these intervals, the radius square function is increasing, then decreasing, then increasing again. The minimum is obtained for a value $z_k \in ((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2})$ such that $\tan(z_k) = z_k$, and for the corresponding radius r_k, we have:

$$r_k > r_1 > r_0,$$

so also in this case the curves $\mathcal{F}_t$ and $\mathcal{S}_t$ do not intersect.

2 Proof of the convergence of the algorithm

For convenience, we first rewrite the algorithm of Section 4, in [1].

ALGORITHM

1. Solve the Time Optimal Control Problem for system 1 and find the minimum time T_1.

2. For $j = 2, ..., N$, check that
   $X_{f,j} \in \mathcal{R}_j(T_1)$
   If this is the case then STOP,
   If this is not the case, find the first j such that $X_{f,j} \notin \mathcal{R}_j(T_1)$; call it $\bar{j}$.

3. Find the smallest $T > T_1$ such that
   $X_{f,\bar{j}} \in \mathcal{R}_{\bar{j}}(T)$

4. Set $T_1 = T$, and exchange 1 with $\bar{j}$.

5. Go back to step 2.

Assume, by the way of contradiction, that the algorithm does not end in a finite number of steps, and consider the sequence of the T_1's, say $T_{1,k} := T_k$. Since this is monotonic and bounded it must have a limit point T^*. Moreover, if the algorithm does not end in a finite number of steps, there must exist a system, let's denote it by $\bar{j} \in \{1, \ldots, N\}$, and two subsequences of times T_{s_k} and T_{i_k}, such that:

$$T_{s_k} < T_{i_k} < T_{s_{k+1}} < T_{i_{k+1}}, \quad X_{f,\bar{j}} \in \mathcal{R}_{\bar{j}}(T_{s_k}),\, X_{f,\bar{j}} \notin \mathcal{R}_{\bar{j}}(T_{i_k}), \tag{34}$$

that is $X_{f,\bar{j}}$ goes inside and outside the reachable set an infinite number of times.

Denote by $P_t = e^{-i\frac{t}{2}} P_{f,\bar{j}}$, and by Q_t the points at the intersection between the curve $\mathcal{F}_{\bar{j},t}$ and the circle $e^{-i\frac{t}{2}} P_{f,\bar{j}}$. Moreover, given a point Q in the circle $e^{-i\frac{t}{2}} P_{f,\bar{j}}$ denote by $p(Q)$ its phase which we assume to be in the interval $[-\pi, \pi)$. We must have:

$$P_{s_k} \in \pi(\mathcal{R}_{\bar{j},U}(T_{s_k})), \quad P_{i_k} \notin \pi(\mathcal{R}_{\bar{j},U}(T_{i_k})). \tag{35}$$

Since the sequence T_k converge to T^*, if we denote by $P^* = e^{-i\frac{T^*}{2}} P_{f,\bar{j}}$, we must have that all four sequences P_{s_k}, P_{i_k}, Q_{s_k}, and Q_{i_k} converge to P^*. Since as t increases the points $e^{-i\frac{t}{2}} P_{f,\bar{j}}$ move clockwise on the circle, while the curve $\mathcal{F}_{\bar{j},t}$ moves from right to left, the point P^* must be either in the third or the the fourth quadrant, because otherwise the interval of time between an entrance in the reachable set of the point P_t and the exit (if any) will increase and cannot go to zero.[5]

From equation (35), we get that:

$$p(Q_{s_k}) < p(P_{s_k}), \quad \text{and} \quad p(Q_{i_k}) > p(P_{i_k}). \tag{36}$$

Let $G(t) = p(P_t) - p(Q_t)$. This function, locally near T^* is analytic, being the sum of two analytic functions of t. This is clear for the points $p(P_t)$, while for the points Q_t it can be proved using the Implicit Mapping Theorem, since these points are the intersections of the circle $e^{-i\frac{t}{2}} P_{f,\bar{j}}$ with the curves $\mathcal{F}_{\bar{j},t}$.

From equation (36), we have:

$$G(s_k) > 0, \quad \text{and} \quad G(i_k) < 0,$$

so in the interval $[T_{s_k}, T_{i_k}]$, the function has a zero, thus there exist $T_{l_k} \in (T_{s_k}, T_{i_k})$, with $G(T_{l_k}) = 0$. But T_{l_k} converges to T^*, and the zeroes of an analytic nonzero function can't have accumulation points. So this gives the desired contradiction and thus the algorithm ends in a finite number of steps.

References

[1] F. Albertini and D. D'Alessandro, Time Optimal Simultaneous Control of Two Level Quantum Systems, *Automatica*, 2016.

[5]More specifically, if P^* is in the first or second quadrant, then the points P_{s_k}, which are in the reachable sets, must converge to P^* from the right, while the points P_{i_k}, which are not in the reachable sets, must converge to P^* from the left. Thus we must have that $p(P_{s_k}) < p(P^*) < p(P_{i_k})$. We have $p(P_{s_k}) = -\frac{T_{s_k}}{2} + p(P_{f,\bar{j}}) + 2L\pi$, for some $L \in \mathbb{Z}$, and $p(P_{i_k}) = -\frac{T_{i_k}}{2} + p(P_{f,\bar{j}}) + 2L'\pi$ for some $L' \in \mathbb{Z}$. Since $|T_{s_k} - T_{i_k}| \to 0$, it must be $L = L'$. But this implies $p(P_{s_k}) > p(P_{i_k})$, which contradicts the fact that the points P_{s_k} is on the right of the point P_{i_k}. So P^* must be either in the third or the the fourth quadrant.