MATH 317 Test # 2 Solutions.

Pr 1 Assume $\vec{v} \in \mathrm{span}(S)$

then

$$\mathrm{span}(S) \subseteq \mathrm{span}(S \cup \{\vec{v}\})$$

is obvious because $S \subseteq S \cup \{\vec{v}\}$

viceversa let $\vec{x}$ be an element of $\mathrm{span}(S \cup \vec{v})$, then, by definition,

$$\vec{x} = \sum_j a_j \vec{s}_j + b \vec{v}$$

for some elements $\vec{s}_j$ of S and with coefficients a_j and b. Since $\vec{v} \in \mathrm{span}$

$\vec{v} = \sum_k c_k \vec{s}_k$ with elements $\vec{s}_k \in S$ and coefficients c_k

therefore

$$\vec{x} = \sum_j a_j \vec{s}_j + b\left(\sum_k c_k \vec{s}_k\right) =$$

$$= \sum_j a_j \vec{s}_j + \sum_k (b c_k) \vec{s}_k$$

therefore $\vec{x} \in \mathrm{span}(S)$.

this shows

$$\mathrm{span}(S) \subseteq \mathrm{span}(S \cup \{\vec{v}\})$$

and concludes the proof that

$$\vec{v} \in \mathrm{span}(S) \implies \mathrm{span}(S) = \mathrm{span}(S \cup \{\vec{v}$$

To prove that

$$\mathrm{span}(S) = \mathrm{span}(S \cup \{\vec{v}\}) \implies \vec{v} \in \mathrm{span}(S)$$

notice that

$$\vec{v} \in \left(S \cup \{\vec{v}\}\right) \subseteq \operatorname{span}\left(S \cup \{\vec{v}\}\right) = \operatorname{span}(S)$$

where in the last equality we used the assumption

Pr 2

$$\det\left(2A^{-1}B^{T}C\right) = 2^{3}\det(A^{-1})\det(B^{T})\det(C) =$$

$$8\,\frac{1}{\det(A)}\det(B)\det(C) = \frac{8}{4}\cdot 8\cdot 6 = 96$$

Pr 3 Characteristic polynomial

$$p_A(\lambda) = \det\begin{pmatrix} 4-\lambda & 1 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (3-\lambda)\left[(1-\lambda)(4-\lambda) + \right.$$

$$=$$

$$(3-\lambda)[4-\lambda-4\lambda+\lambda^2+2\} =$$

$$(3-\lambda)[\lambda^2-5\lambda+6\} = (3-\lambda)(\lambda-3)(\lambda-2)$$

eigenvalues

$\lambda = 2 \qquad n_1 = 1$ ← algebraic multiplicities

$\lambda = 3 \qquad n_2 = 2$

Find eigenvectors.

$\lambda = 2$

$$(A-2I) = \begin{pmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A-2I)X = 0 \qquad X = \begin{pmatrix} \frac{1}{2} \\ -1 \\ 0 \end{pmatrix}$$

$\lambda = 3$ 113

$$(A - 3I) = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A-3I)X = 0 \qquad X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

linearly independent

$$P = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left(X \;\; X_1 \;\; X_2 \right).$$

$$P^{-1} A P = \begin{bmatrix} 2 & & 0 \\ & 3 & \\ 0 & & 3 \end{bmatrix}$$

Pr 4

a) No it is not a subspace.

2 which is $2\cdot 1$ is not there

It is not closed under multiplication by a scalar.

b) $B_1 = \{1, 1+x^3, 1+x^2+2x\}$

(eliminate the lin. dep. element $3+2x^2$)

$B_2 = \{1, x^3, 1+x^2+2x\}$

c) dimension is 3

d) Complete B_2 by adding x^2

$B = \{1, x^3, x^2, 1+x^2+2x\}$ is a basis of P_3

Pr 5

$$A = a_1\begin{pmatrix}1 & 1\\ 0 & 0\\ 0 & 0\end{pmatrix} + a_2\begin{pmatrix}2 & 1\\ 0 & 0\\ 0 & 0\end{pmatrix} + a_3\begin{pmatrix}0 & 0\\ 1 & 1\\ 0 & 0\end{pmatrix} + a_4\begin{pmatrix}0 & 0\\ 3 & 0\\ 0 & 0\end{pmatrix}$$

$$+ a_5\begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 1\end{pmatrix} + a_6\begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & -1\end{pmatrix} = \begin{pmatrix}2 & 3\\ 1 & 2\\ 0 & 1\end{pmatrix}$$

$$\left.\begin{aligned} a_1 + 2a_2 &= 2 \\ a_1 + a_2 &= 3 \end{aligned}\right\} \Rightarrow a_2 = -1 \quad a_1 = 4$$

$$\left.\begin{aligned} a_3 + 3a_4 &= 1 \\ a_3 &= 2 \end{aligned}\right\} \Rightarrow a_3 = 2 \quad a_4 = -\frac{1}{3}$$

$$\left.\begin{aligned} a_5 + a_6 &= 0 \\ a_5 - a_6 &= 1 \end{aligned}\right\} \Rightarrow a_5 = \frac{1}{2} \quad a_6 = -\frac{1}{2}$$

$$[A]_B = \begin{bmatrix} 4 \\ -1 \\ 2 \\ -\frac{1}{3} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

<u>Prob 6</u> Calculation of P_{BC} by direct method.

Define $C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $C_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$C_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1\, C_1 + 0\, C_2 + 0\, C_3$$

$$\begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix} = -4\, C_1 + 4\, C_2 + 2\, C_3$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\, C_1 - C_2 - C_3$$

So $P_{BC} = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 4 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

Using standard basis

$P_{BS} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ $\qquad P_{CS} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$P_{BC} = \quad P_{SC} P_{BS} = (P_{CS})^{-1} P_{BS}$

$P_{CS}^{-1} \rightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Longrightarrow$$

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow$$

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$P_{CS}^{-1} =$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$P_{CS}^{-1} P_{BS} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 4 & -1 \\ 0 & 2 & -1 \end{pmatrix}$$