MATH 317 Test 3 Solutions

Problem 1

a) Assume $\vec{x} \in \mathrm{Ker}(L_1)$ then

$$L_1(\vec{x}) = 0.$$

$$(L_2 \circ L_1)(\vec{x}) = L_2(L_1(\vec{x})) = L_2(\vec{0}) = 0 \Rightarrow$$

$$\Rightarrow \vec{x} \in \mathrm{Ker}\, L_2 \circ L_1$$

b) Assume $\vec{y} \in$ range $L_2 \circ L_1$. Then there exists $\vec{x} \in V$ such that

$L_2(L_1(\vec{x})) = \vec{y}$ but this implies that there exists $\vec{w} \in W$, that is $\vec{w} = L_1(\vec{x})$ such that $L_2(\vec{w}) = \vec{y} \Rightarrow \vec{y} \in$ range L_2 ∎

Problem 2

$L(1) = 1$ $\quad$ $L(1+x) = 1 + x + 1 = 2 + x$

$$L(1-x^2) = 1-x^2-2x$$

$$[L(1)]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad [L(1+x)]_B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$[L(1-x^2)]_B = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$A_{BB} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 3

$$A_{SS} = P_{BS} A_{BB} P_{SB}$$

$$P_{BS} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Find $P_{SB} = P_{BS}^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

P_{SB}

$$A_{SS} = \overset{P_{BS}}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}} \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 4

Consider standard basis in $M_{2,2}$.

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

$$L\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$L\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$L\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$L\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A_{SS} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

a) range corresponds to span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} =$

range = span$\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)$

b) $\mathrm{Ker}(A_{ES}) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

corresponds to

span$\left(\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right)$.

c) $\dim(\mathrm{Ker}(L)) = 2$ $\quad$ $\dim(\mathrm{range}(L)) = 2$

$2 + 2 = 4 = \dim M_{2,2}$

d) No $\mathrm{Ker}(L) \neq \{\vec{0}\}$

Problem 5

$$A_{SS} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\det(A_{SS} - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{pmatrix} =$$

$$(1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1-\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$(1-\lambda)\left[(1-\lambda)^3 - (1-\lambda)\right\} + 1\left(1 - (1-\lambda)^2\right) =$$

$$(1-\lambda)^2\left[(1-\lambda)^2 - 1\right\} + \left(1 - (1-\lambda)^2\right)$$

$$\left[(1-\lambda)^2-1\right\}\left[(1-\lambda)^2-1\right\}=$$

$$\left[(1-\lambda)^2-1\right\}^2=0 \iff (1-\lambda)^2-1=0$$

$$(1-\lambda)^2-1=\lambda^2-2\lambda+1-1=0 \quad \lambda(\lambda-2)$$

characteristic polynomial.

$$p(\lambda)=\lambda^2(\lambda-2)^2$$

eigenvalues $\lambda_1=0$ $\lambda_2=2$

a) Eigenspace corresponding to $\lambda_1=0$ =

$$=\ker = \text{span}\left(\begin{pmatrix}1&0\\-1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&-1\end{pmatrix}\right)$$

Eigenspace corresponding to $\lambda_2=2$.

$$A_{SS}-2I=\begin{pmatrix}-1&0&1&0\\0&-1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$x_3 = x_1$ $\quad$ $x_2 = x_4$ $\quad$ using standard basis.

① eigenspace is given by.

$$\text{span}\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right).$$

c) Yes it is diagonalizable because there are 4 linearly independent eigenvectors.

<u>Problem 6</u> $\quad$ Complete with another vector.

$$\vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{u}_1, \vec{u}_2, \vec{u}_3 \text{ lin. ind.}$$

Gram-Schmidt to get orthogonal.

11)

$\vec{v}_1 = \vec{u}_1$

$\vec{v}_2 = \vec{u}_2$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle \vec{v}_1}{\langle \vec{v}_1, \vec{v}_1 \rangle} - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle \vec{v}_2}{\langle \vec{v}_2, \vec{v}_2 \rangle} =$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{1+4+1} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - 0 =$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{5}{6} \\ -\frac{1}{3} \\ +\frac{1}{6} \end{pmatrix}$$

Set $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} \frac{5}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \end{pmatrix}$