Summary of More Recent Research

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My current area of research is in control and analysis of quantum mechanical systems. In this study, I have applied and enhanced tools of geometric control theory and investigated models and mathematical problems in quantum information theory. Previous research has focused on general nonlinear control theory, optimal control of linear and nonlinear systems and applications of ergodic theory to the control and analysis of fluid mixing. In this summary, I will not consider contributions in these previous areas of research as I will focus on more recent research. Numbered references refer to publications listed in the ‘Publications’ section. References denoted by a letter are listed in the references section.

Methodologies for quantum control based on decompositions of Lie groups

Decompositions of Lie groups are of great interest in the theory of quantum control and quantum information. They allow one to decompose the task of controlling the state of a quantum system into simpler sub-tasks. These decompositions also reveal several features of quantum dynamics concerning entanglement and locality. Moreover they are the starting point for the design of the so-called quantum circuits, that is, cascades of unitary quantum evolutions from a certain set (quantum gates), which perform prescribed transformations of the quantum state (computations) (see, e.g., [A]). One of the first studies in this direction was [24] where I provided a very explicit method for the control of a spin $\frac{1}{2}$ particle and two interacting spin $\frac{1}{2}$ particles. This method of design can take into account an arbitrary bound on the control. Other algorithms for control which use Lie group decompositions are given in [19], [28]. For the case of the Lie group $SU(2)$, which is the Lie group of interest in the analysis of a spin $\frac{1}{2}$, I have provided a switching control algorithm in [21] which achieves the target with the minimum number of switches. These results are of general interest in the theory of Lie groups. In [B], [C] and my previous work [26] the problem of uniform finite generation of Lie groups was considered. Given a set of generators $A$ of a Lie algebra $L$, the Lie group associated with $L$, is said to be uniformly finitely generated by $A$, if every element $X$ of the Lie group can be written as $X = \prod_{j=1}^{N} e^{F_j t_j}$ with $F_j \in A$, $t_j \in \mathbb{R}$ and the number $N$ is uniformly bounded as $X$ varies in the Lie group. The maximum value of $N$ is called the order of generation of the Lie group with respect to $A$. The results of [21] go beyond what was known before in that, given a set of generators and a value $X$ in the Lie group $SU(2)$ (or $SO(3)$) the exact minimum number of factors is explicitly calculated as a function of $X$.

When dealing with multipartite quantum systems, it is natural to look for decompositions of the Lie group of evolutions, $U(n)$, in terms of evolutions on the single quantum subsystems. A first study given in [D] proposed the Concurrence Canonical Decomposition (CCD). The concurrence is a measure of quantum entanglement, that is, a function of the quantum state, satisfying certain axioms [E] which measures the quantity of entanglement of the
state. The CCD is a way to decompose the evolution of the quantum state in a part which modifies the concurrence (and therefore entanglement) and a part that does not. It applies to systems of \(N\) qubits (two level quantum systems) and it is a tool to study entanglement dynamics. In [11], I have greatly generalized the CCD introducing a new type of decomposition which I called the \textit{Odd-Even Decomposition} (OED). The OED is a way to decompose the dynamics of a general multipartite quantum system, with arbitrary dimensions, using Cartan decompositions [F], defined for the single subsystems. The CCD is obtained as a special case when all the subsystems have dimension 2 and only a special type of Cartan decomposition is performed on each subsystem. As discussed in my book [1], the OED decomposition is related to \textit{generalized concurrences}, introduced in [G], which have a role in the analysis and detection of entanglement. The paper [11] contains several further results. It establishes a decomposition of the dynamics of quantum systems in terms of time symmetries, generalizing the results of [H] for spin \(\frac{1}{2}\)'s to general spins and it links \textit{quantum symmetries} [I] to the \textit{Cartan involutions} [F] used in the classification of symmetric spaces.

Applying Lie group decompositions in a recursive manner, one can obtain more useful decompositions. This has led to a large effort by researchers to give algorithms for the \textit{recursive} decomposition of Lie groups and in particular of the unitary group \(U(n)\). One such decomposition was given by N. Khaneja and S. Glaser in [J] for a quantum system of \(N\) qubits. Another decomposition was given by my R. Romano and myself in [12]. This line of research has lead to the recent paper [9], co-authored by M. Dagli, J.Smith and myself. In this paper, we recognized a link between recursive Lie algebra decompositions and \textit{Lie algebra gradings} and the fact that Lie algebra gradings can be obtained by combining various decompositions. Using these ideas, it is possible to design a virtually unbounded number of algorithms for recursive decompositions and I showed that the papers [J] and [12] are special cases of this general procedure. In constructing recursive decompositions one uses several types of decompositions and, in that, the OED above discussed is a useful tool. In fact, an example of a new algorithm for the design of recursive decompositions using OED was given in [9].

\textbf{Identification of the dynamics of quantum spin systems}

In [20] F. Albertini and I started a research on the identification problem for spin systems. This problem was motivated by early results on the iso-spectrality of spin Hamiltonians in magnetic molecules [K] and may be described as follows. Assume a network of interacting spins is given as, for example, in a molecule. These spins interact with each-other and with an external control electro-magnetic field. Moreover, the total spin in some direction is measured. The model is characterized by several parameters, including the number of spin particles, the value of their spin, the strength of the interaction between the various spins, the strength of the interaction with the control electro-magnetic field, the initial state of the system. Assume, we perform black-box type of experiments by varying the control and measuring the total spin. We would like to know to what extent we can distinguish different models with this type of technique. The mathematical problem is to classify all the models that are indistinguishable with these experiments. The problem was solved in [20] for networks of spin \(\frac{1}{2}\) and then this result was extended in [18] by considering the possibility
of performing multiple measurements during the experiment and therefore incorporating the back-action of the measurement in the dynamics. The problem was then solved for the case of two spin 1’s in [17] and in full generality in [8]. In the latter work, no assumption is made on the number of spin particles nor on their values (which could be different from one another). Moreover in this work a method based on concepts of Lie algebra theory is introduced to tackle this type of equivalence problems for other models. Instrumental to the proof of the main result in [8] is the OED decomposition introduced in [11]. Moreover, the paper [8] contains several proofs for results on subalgebras of the classical Lie algebras which are of independent interest.

**Controllability and dynamical analysis**

The application of geometric control theory results [L] solved the problem of determining the set of possible evolutions of finite dimensional quantum systems (see, e.g., [Q]). A quantum system is controllable and therefore the set of all the possible evolutions is the whole Lie group $U(n)$ (or $SU(n)$) if the Lie algebra generated by all the possible Hamiltonians of the system is the full $u(n)$ or $su(n)$. More, in general, this Lie algebra, called the dynamical Lie algebra, determines the Lie group on which the evolution of the quantum system occurs. Several questions remained however open concerning controllability in terms of state to state transfer. In the paper [23] I defined several physically motivated notions of controllability for quantum systems and gave practical criteria to verify them. These tests involve only linear algebra type of calculations, such as determining the rank and the dimension of the kernel of a linear transformation. One more notion of controllability, indirect controllability, was studied by my Raffaele Romano and myself in [13], [16]. Indirect controllability refers to the situation where the Hamiltonian of a quantum system is given and the control occurs via interaction with an auxiliary quantum system (the probe) whose initial state can be adjusted (at will).

The knowledge of the dynamical Lie algebra $\mathcal{L}$ not only gives information on the controllability of the quantum system under consideration but determines the very structure of the dynamics of the system. One typically has a set of matrices which are a basis of the Lie algebra $\mathcal{L}$ and would like to explore its nature. Levi’s theorem (see, e.g., [M]) implies that $\mathcal{L}$ is the vector space sum of a semisimple Lie algebra and a solvable Lie algebra. Moreover the semisimple part (by definition) can be decomposed into simple Lie algebras. This decomposition determines the structure of the dynamics of the quantum system. Algorithms to calculate such a decomposition exist for general Lie algebras [M] but I have given in [44] simplified algorithms for the special case of interest in quantum control, where $\mathcal{L}$ is a subalgebra of $u(n)$. The analysis of the dynamical Lie algebra provides a powerful approach to the control of quantum systems and several examples of this are given in [44].

**Methods to determine the state of quantum systems**

One important area of study in quantum mechanics deals with determining or estimating the state of a quantum system through the measurement of a quorum of observables. In control theory, one studies to what extent it is possible to determine the initial state of a system by
a combined action of dynamics and measurement. Such a property is called observability. In [22], I studied the observability of quantum systems and gave definitions and tests for it. I showed how quantum dynamics can in general decomposed in an observable and unobservable part. This study, mostly concerned the standard Von Neumann-Lüders type of measurement of quantum mechanics. However, in [15], R. Romano and I extended it to include more general measurements (cf., e.g., [N]). The paper [15] emphasizes the treatment of indirect measurement in which a probe interacts with the system to be measured. One of the results in [15] characterizes the state of the probe which will minimally disturb the system upon measurement. I also gave explicit formulas for the calculation of the quantum state from measurement results and described the design of observers for quantum systems. These results on one hand are of fundamental value in the description of structural properties of quantum systems and on the other hand provide a system theoretic alternative to the method of quantum state tomography [O] commonly used for quantum state determination. In the fourth chapter of my book [1] I summarized some of these contributions and relate them to quantum state tomographic methods.

Quantum walks

Quantum walks are the quantum counterpart of random walks and are amenable of the same important applications, in particular, as computational tools. Moreover it has been shown (see, e.g., [P]) that in some cases they outperform random walks. In [10] F. Albertini, G. Parlangeli and I started a study of quantum walks on graphs [P] from the point of view of control theory. In the model I considered, the evolution of the walk at each step can change, and can be controlled. This can be obtained with only minor modifications in the existing experimental proposals and it gives several advantages. In [10] I considered the simplest case of a random walk on a cycle and proved two main results: 1) I described the set of all the states that can be achieved for this system; 2) I proved that one of these states is such that the probability is uniformly distributed among the nodes of the associated graph. This is in contrast with the case of standard quantum random walks, where the dynamics is the same at every step. This fact has important applications if, for example, we are trying to use the walk as a random number generator in randomized algorithms. Further results on the controllability of this model were given in [44].

Overall, I believe this area of research is very interesting. In the future I am planning to consider various types of walks, with different graphs, and further explore the interplay between their properties as control systems and their applications as computational tools.

References


## 1 Publications

(Several Preprints are Available at http://www.public.iastate.edu/daless/ or xxx.lanl.gov/quant-ph)

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[53] F. Albertini and D. D’Alessandro, Observability, measurement and parameter identification of quantum mechanical systems, in Proceedings of the 42-nd Conference on Decision and Control


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