MATH 317 Section A, Practice Final

PART I (75 Points)

1. Give the definition of the zero element $\vec{0}$ of a vector space $\mathcal{V}$ and give an example of the zero of a vector space $\mathcal{V}$ where $\mathcal{V}$ is **not** $\mathbf{R}^n$.

2. Give an example of a $2-$ dimensional subspace of the vector space of 2×2 matrices, $M_{2,2}$.

3. Give the definition of eigenspace of a matrix A associated to an eigenvalue λ. Give an example of a 2×2 matrix A with an eigenvalue equal to 1 and describe the corresponding eigenspace.

4. Give the definition of linear independent vectors. Give an example of three vectors in $M_{2,2}$ (space of 2×2 matrices), all different that are **not** linearly independent.

5. Give two different bases in the space $M_{3,2}$ of 3×2 matrices. What is the dimension of this vector space?

6. Let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ a set of linearly independent vectors of a vector space $\mathcal{V}$. If $dim(\mathcal{V}) = n$ what can one say about the linear independence of $S_1 = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n, \vec{w}\}$, for any vector $\vec{w} \in \mathcal{V}$ and of $S_2 = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$, for $k < n$?

7. Consider the two ordered bases of $\mathcal{P}_2$,

$$B = \{1, 1 + x, x - x^2\},$$

and

$$C = \{1, x^2, 1 - x\}.$$

Find the transition matrix which allows to obtain coordinates in the basis C from coordinates in the basis B, i.e. P_{BC}. If a vector has coordinates $[1, 2, 0]$ in the basis B, what are its coordinates in the basis C?

8. Give the definition of Kernel and Range of a linear transformation. Give an example of a linear transformation $\mathcal{P}_2 \to \mathcal{P}_2$, where $\mathcal{P}_2$ is the vector space of polynomials of degree up to 2, for which the polynomial $1 + x$ is **not** in the Kernel.

9. Consider the linear transformation $L : \mathcal{P}_1 \to \mathcal{P}_1$, where $\mathcal{P}_1$ is the vector space of polynomials of degree up to 1, and L is given by

$$L(p) = p + p'.$$

Fix a basis in $\mathcal{P}_1$ and determine the matrix representation of the linear operator L in this basis.

10. Consider the linear operator $L : \mathbf{R}^n \to \mathbf{R}^n$ defined, in the standard basis, by

$$A_S = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Consider the alternative basis

$$B = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \},$$

and give the expression of the same linear operator in the basis B.

11. Give the statement of the dimension theorem and illustrate the theorem with an example of a linear operator with one-dimensional Kernel and one-dimensional range.

12. Give the definition of **one-to-one** transformation and the definition of **onto** transformation. Give the statement of a theorem which relates the fact that a linear transformation is one to one to the Kernel of that linear transformation.

13. Consider the linear operator

$$L : M_{2,2} \to M_{2,2},$$

where $M_{2,2}$ is the space of 2×2 matrices. L defined by

$$L(A) = A - A^T.$$

What are the eigenvalues of this operator? What are the eigenspaces? Is this operator diagonalizable? and why?

14. Give the definition of orthogonal and orthonormal vectors in $\mathbf{R}^n$. Give three examples: 1) Two vectors in $\mathbf{R}^3$ which are not orthogonal 2) Two vectors in $\mathbf{R}^3$ which ar orthogonal but not orthonormal. 3) Two vectors in $\mathbf{R}^3$ which are orthonormal.

15. Give the definition of orthogonal matrix and a 2×2 example different from the identity matrix.

PART II (25 Points)

Problem 1 (15 points) Use the Gram-Schmidt process to find an orthogonal basis of

$$span\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\}$$

Problem 2 (10 points) Find the Least Squares best solution of the inconsistent system

$$AX = B,$$

where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$