PART I (75 Points)

1. Give the definition of the zero element $\vec{0}$ of a vector space $V$ and give an example of the zero of a vector space $V$ where $V$ is not $\mathbb{R}^n$.

2. Give an example of a 2-dimensional subspace of the vector space of $2 \times 2$ matrices, $M_{2,2}$.

3. Give the definition of eigenspace of a matrix $A$ associated to an eigenvalue $\lambda$. Give an example of a $2 \times 2$ matrix $A$ with an eigenvalue equal to 1 and describe the corresponding eigenspace.

4. Give the definition of linear independent vectors. Give an example of three vectors in $M_{2,2}$ (space of $2 \times 2$ matrices), all different that are not linearly independent.

5. Give two different bases in the space $M_{3,2}$ of $3 \times 2$ matrices. What is the dimension of this vector space?

6. Let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ a set of linearly independent vectors of a vector space $V$. If $dim(V) = n$ what can one say about the linear independence of $S_1 = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n, \vec{w}\}$, for any vector $\vec{w} \in V$ and of $S_2 = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$, for $k < n$?

7. Consider the two ordered bases of $\mathcal{P}_2$,

$$B = \{1, 1 + x, x - x^2\},$$

and

$$C = \{1, x^2, 1 - x\}.$$  

Find the transition matrix which allows to obtain coordinates in the basis $C$ from coordinates in the basis $B$, i.e. $P_{BC}$. If a vector has coordinates $[1, 2, 0]$ in the basis $B$, what are its coordinates in the basis $C$?

8. Give the definition of Kernel and Range of a linear transformation. Give an example of a linear transformation $\mathcal{P}_2 \to \mathcal{P}_2$, where $\mathcal{P}_2$ is the vector space of polynomials of degree up to 2, for which the polynomial $1 + x$ is not in the Kernel.
9. Consider the linear transformation \( L : \mathcal{P}_1 \rightarrow \mathcal{P}_1 \), where \( \mathcal{P}_1 \) is the vector space of polynomials of degree up to 1, and \( L \) is given by
\[
L(p) = p + p'.
\]
Fix a basis in \( \mathcal{P}_1 \) and determine the matrix representation of the linear operator \( L \) in this basis.

10. Consider the linear operator \( L : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined, in the standard basis, by
\[
A_S = \begin{pmatrix}
1 & 0 & -1 \\
1 & 0 & 1 \\
2 & 1 & 1
\end{pmatrix}.
\]
Consider the alternative basis
\[
B = \left\{ \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
-1
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix} \right\},
\]
and give the expression of the same linear operator in the basis \( B \).

11. Give the statement of the dimension theorem and illustrate the theorem with an example of a linear operator with one-dimensional Kernel and one-dimensional range.

12. Give the definition of one-to-one transformation and the definition of onto transformation. Give the statement of a theorem which relates the fact that a linear transformation is one to one to the Kernel of that linear transformation.

13. Consider the linear operator
\[
L : M_{2,2} \rightarrow M_{2,2},
\]
where \( M_{2,2} \) is the space of \( 2 \times 2 \) matrices. \( L \) defined by
\[
L(A) = A - A^T.
\]
What are the eigenvalues of this operator? What are the eigenspaces? Is this operator diagonalizable? and why?

14. Give the definition of orthogonal and orthonormal vectors in \( \mathbb{R}^n \). Give three examples: 1) Two vectors in \( \mathbb{R}^3 \) which are not orthogonal 2) Two vectors in \( \mathbb{R}^3 \) which are orthogonal but not orthonormal. 3) Two vectors in \( \mathbb{R}^3 \) which are orthonormal.
15. Give the definition of orthogonal matrix and a $2 \times 2$ example different from the identity matrix.

**PART II (25 Points)**

**Problem 1** (15 points) Use the Gram-Schmidt process to find an orthogonal basis of 

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

**Problem 2** (10 points) Find the Least Squares best solution of the inconsistent system 

$$AX = B,$$

where 

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$