Problem 1. (20 points)
Assume $A$ is an $n \times n$ nonsingular matrix. Which of the following statements must be false, which one may be true and which one must be true. Justify your answer and in the case where it may be true give an example.

1. $A$ is an upper triangular matrix.
2. The reduced row echelon form of $A$ is the identity.
3. The standard vector
   \[ e_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \]
   is a solution of the system $AX = 0$
4. $\det A = 0$

Problem 2 (20 points) Consider the matrix
\[ A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \]

a) Calculate the determinant and conclude that $A$ has an inverse.
b) Use Gauss-Jordan elimination to find $A^{-1}$
c) Use the inverse to calculate the solution of the system
\[ AX = B \]
where $B := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Problem 3 (20 points) Calculate the determinant of the following matrix by using expansion along the second column
\[ A := \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}. \]

Problem 4 (20 points)
a) Diagonalize the following $2 \times 2$ matrix
\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \]
b) Calculate $A^6$ using the diagonalization of $A$

\[ A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix} \]

e) Show that the following $3 \times 3$ matrix is not diagonalizable.

**Problem 5** (20 points) Consider a diagonal matrix $A$, $n \times n$.

a) Prove that $A$ is nonsingular if and only if all the elements on the diagonal are different from zero.

b) Prove that $A^{-1} = B$ where $B$ is also diagonal and $b_{ii} = \frac{1}{a_{ii}}$, $i = 1, ..., n.$