Practice Test #1 Solutions. 1

Pr 1)

a

1) $AB = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 6 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 8 & 1 \\ 6 & -8 \end{pmatrix}$

2) $(AB)^T = \begin{pmatrix} 1 & 8 & 6 \\ 1 & 1 & -8 \end{pmatrix}$

3) $A\vec{v} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 17 \end{pmatrix}$

$A\vec{v} \cdot \vec{v} = \begin{pmatrix} -2 \\ 1 \\ 17 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0(-2) + (1)(1) + 34 = 35$

b) $(AB+BA)^T = (AB)^T + (BA)^T =$

$B^T A^T + A^T B^T = BA + AB =$

$AB + BA \quad \checkmark$

$(AB-BA)^T = (AB)^T - (BA)^T =$

$B^T A^T - A^T B^T = (-B)(-A) - (-A)(-B) =$

$BA - AB = -(AB - BA) \quad \checkmark$

Problem 2). Consider augmented matrix

$$\begin{bmatrix} 2 & 4 & 0 & 1 & 4 \\ 1 & 2 & 1 & 4 & 1 \end{bmatrix}$$

I) $(1) \leftarrow \frac{1}{2}(1)$

$$\begin{bmatrix} 1 & 2 & 0 & \frac{1}{2} & 2 \\ 1 & 2 & 1 & 4 & 1 \end{bmatrix}$$

II) $(2) \leftarrow (2) - (1)$

$$\begin{bmatrix} 1 & 2 & 0 & \frac{1}{2} & 2 \\ 0 & 0 & 1 & \frac{7}{2} & -1 \end{bmatrix}$$

$x_1 \quad x_3$ dependent variables.

$x_2 \quad x_4$ independent variables.

$x_2 = \lambda \qquad x_4 = \mu.$

$x_1 + 2\lambda + \frac{1}{2}\mu = 2 \qquad x_1 = 2 - 2\lambda - \frac{1}{2}\mu.$

$x_3 + \frac{7}{2}\mu = -1 \qquad x_3 = -1 - \frac{7}{2}\mu.$

Problem 3)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$\underline{\text{II}}$)

$$(2) \leftarrow (2) - 2(1)$$

$$\begin{array}{ccc} 2 & 0 & 1 \; - \\ 2 & 2 & 2 \\ \hline 0 & -2 & -1 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$\underline{\text{II}}$

$$(3) \leftarrow (3) - (1)$$

$$\begin{array}{ccc} 1 & -1 & 0 \\ 1 & 1 & 1 \\ \hline 0 & -2 & -1 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{pmatrix}$$

I $\quad (2) \leftarrow -\frac{1}{2}(2)$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & -2 & -1 \end{pmatrix}$$

II

$$(3) \leftarrow (3) - [-2](2)$$

$$\begin{array}{cccc} 0 & -2 & -1 & - \\ 0 & -2 & -1 & \\ \hline 0 & 0 & 0 & \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$(1) \leftarrow (1) - (2)$$

$$\begin{array}{cccc} 1 & 1 & 1 & - \\ 0 & 1 & \frac{1}{2} & \\ \hline 1 & 0 & \frac{1}{2} & \end{array}$$

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$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

Pr 4) By definition

$$(1) \quad \mathrm{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a}$$

Using ~~the~~ elementary properties of the norm we obtain

$$(2) \quad \left\| \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a} \right\| = \left| \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \right| \|\vec{a}\| = \frac{|\vec{a} \cdot \vec{b}|}{\|\vec{a}\|}$$

Moreover from Cauchy-Schwartz inequality we obtain

$$(3) \quad \frac{|\vec{a}\cdot\vec{b}|}{\|\vec{a}\|} \leq \frac{\cancel{\|\vec{a}\|}\,\|\vec{b}\|}{\cancel{\|\vec{a}\|}} = \|\vec{b}\|$$

Combining (1) (2) and (3) we obtain

$$(4) \quad \|\operatorname{Proj}_{\vec{a}} \vec{b}\| \leq \|\vec{b}\|$$

as desired.

The proof that equality holds in (4) if and only if $\vec{b}$ is parallel to $\vec{a}$ is divided in two parts. First assume that $\vec{b}$ is parallel to $\vec{a}$. We have

$$\vec{b} = c\,\vec{a} \qquad \text{for some scalar } c$$

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We have

$$\mathrm{Proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a} = c\,\frac{\vec{a}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a} = c\vec{a} = \vec{b}$$

and therefore $\|\mathrm{Proj}_{\vec{a}}\vec{b}\| = \|\vec{b}\|$.

Then assume $\|\mathrm{Proj}_{\vec{a}}\vec{b}\| = \|\vec{b}\|$. We have

$$\frac{|\vec{a}\cdot\vec{b}|}{\|\vec{a}\|} = \|\vec{b}\|$$

and therefore

$$|\vec{a}\cdot\vec{b}| = \|\vec{b}\|\|\vec{a}\|$$

From Theorem 1.5 this implies $\vec{b}$ parallel to $\vec{a}$