Range and Kernel of a general linear Transformation.

<u>Problem</u> Given a linear Transformation $T: V \to W$ where V and W are vector spaces of dimensions n and m, respectively how do we find $\ker T$ and Range of T?

<u>Theorem</u> Fix a basis B for V and a basis C for W.

$$B = \{\vec{u}_1, \dots \vec{u}_n\}.$$

$$C = \{\vec{v}_1, \dots, \vec{v}_m\}$$

Consider the matrix

$$A_{BC} = [[T\vec{u}_1]_C \; [T\vec{u}_2]_C \dots [T\vec{u}_n]_C]$$

Then for every $\vec{x} \in V$,

$$[T\vec{x}]_C = A_{BC}[\vec{x}]_B.$$

This theorem reduces the analysis of every linear transformation on any space to the analysis of a matrix transformation A_{BC} from $\mathbb{R}^n$ to $\mathbb{R}^m$.

It says that the following commutative diagram holds.

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "{[\ ]_B}"'] & W \arrow[d, "{[\ ]_C}"] \\
\mathbb{R}^n \arrow[r, "A_{BC}"] & \mathbb{R}^m
\end{tikzcd}$$

p^2

EXAMPLE 1

Consider $T: P_2 \to \mathbb{R}^2$

$$T(p) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}$$

Fix basis in P_2, $B = \{1, t, t^2\}$.

Fix basis in $\mathbb{R}^2$ $C = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

The matrix

$$A_{BC} = \left[[T(1)]_C \ [T(t)]_C \ [T(t^2)]_C \right] =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

For a general $p(t) = a + bt + ct^2$

$$T(p) = \begin{bmatrix} a \\ a \end{bmatrix} \qquad [T(p)]_C = \begin{bmatrix} a \\ a \end{bmatrix}$$

$$[v]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

EXAMPLE 2

Consider $T : M_{2,2} \to M_{2,2}$.

defined as

$$T(A) = 2A - A^T$$

this is a linear transformation

$$B = \left\{ \underset{A_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{A_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{A_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{A_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}.$$

$$C = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

calculate $\quad\quad\quad$ 113

$$T(A_1) = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T(A_2) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

$$T(A_3) = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & +1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$$

$$T(A_3) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[T(A_1)]_C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad [T(A_3)]_C = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$[T(A_2)]_C = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} \qquad [T(A_4)]_C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A_{BC} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix A_{BC} is used to determine the Kernel and the Range of T

To do this we need to introduce the Inverse coordinatization, i.e., inverse coordinate map. - Recall coordinate map $[\ ]_B$

$$V \xrightarrow{[\ ]_B} \mathbb{R}^n$$

The inverse coordinate map. $[\ ]_B^{-1}$

$$\mathbb{R}^n \xrightarrow{[\ ]_B^{-1}} V$$ it works like.

$(B = \vec{u}_1, \dots \vec{u}_n)$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow x_1\vec{u}_1 + x_2\vec{u}_2 + \dots + x_n\vec{u}_n$$

EXAMPLE

let $B = \{1, 1+t, t^2\}$

consider $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. We have $\left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right]_B^{-1} =$

$$1 + 2(1+t) + 3t^2 \quad ✓$$

ALGORITHM TO Find Kernel of T

1. Find A_{BC}
2. Find Nul A_{BC}
3. Do inverse coordinatization with B

this gives Ker T

EXAMPLE

Recall the linear transformation

$T : P_2 \to \mathbb{R}^2 \qquad T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.

we have found

$$A_{BC} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\text{Nul } A_{BC} = \left\{ \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}, \quad x_2, x_3 \in \mathbb{R} \right\}$$

all these vectors are such that

$$A_{BC} \vec{x} = 0$$

Recall $B = \{1, t, t^2\}$ - Inverse coordinatization gives

$\ker F = \{ x_2 t + x_3 t^2 \mid x_2, x_3 \in \mathbb{R} \}$.

EXAMPLE 2.

Recall the map $T(A) = 2A - A^T$

we had
$$A_{BC} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Nul $A_{BC} = \{ \vec{0} \}$ because the matrix A_{BC} is non singular

therefore $\ker(T) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ only the zero matrix gives "0".

ALGORITHM TO FIND Range of T

① Find A_{BC}

② Find Col(A_{BC})

③ Do inverse coordinatization (with C)

This gives range of T

EXAMPLE.

Consider again

$$T(p) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}.$$

$$A_{BC} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

~~range of T = span$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$~~

$$\text{col } A_{BC} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \qquad \text{Range } T = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

EXAMPLE

consider again

$T(A) = 2A - A^T$

Invers Coordinatization

$\mathrm{Col}\, A_{BC} = \mathbb{R}^4 \longrightarrow$ Range of $T = M_{2,2}$.

EXAMPLE

consider the linear transformation.

$$T(p) = \frac{d}{dt} p(t) + 2p(t)$$

$$T : P_2 \to P_2$$

let $B = C = \{1, t, t^2\}$.

$A_{BC} = ?$

$T(1) = 2$

$T(t) = 1 + 2t$

$T(t^2) = 2t + 2t^2$

$$A_{BC} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$\ker T = \{0\}$.

$\text{Range}\, T = P_2$