SMALL TIME CONTROLLABILITY OF SYSTEMS ON COMPACT LIE GROUPS AND SPIN ANGULAR MOMENTUM

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Abstract

In this paper, we develop some general results on the properties of the reachable sets for right invariant bilinear systems with state varying on compact Lie groups. The main results consist of a characterization of the set of states reachable in arbitrary time from the identity of the group. This, under suitable assumptions, is proved to be a Lie subgroup of the underlying Lie group. We apply these results to the analysis of the controllability of particles with spin. The results are motivated by and generalize the results in [D. D'Alessandro, Topological properties of the reachable sets and the control of quantum bits, *Systems and Control Letters*, 41 (2000) 213-221], where the specific model of a spin $\frac{1}{2}$ particle system in an electro-magnetic field was considered.

Keywords: Control of Quantum Mechanical Systems, Reachable sets, Systems on Lie Groups, Particles with Spin.

I. Introduction

In recent years there has been a large amount of interest in the development and application of techniques from control theory for the manipulation of the state of quantum mechanical systems (see e.g.[1], [2], [3], [4]). In typical laboratory experiments, an electro-magnetic field is used to drive the state of a quantum system to a desired value. The electro-magnetic field is seen as a control and classical issues of control theory such as analysis of controllability and development of methods for control have a natural physical interpretation in this setting. As an application of the results in this paper, we will consider the simple but important example of a particle with spin angular momentum immersed in an electro-magnetic field and perform an analysis of the controllability of this system. No restriction will be placed on the value of the spin of the particle so that the systems considered include for example both Helium ^3He and ^4He molecules as well as an electron and a proton. In physical situations a time varying electro-magnetic field is used to induce a rotation and an analysis of the controllability for this kind of systems tells us what rotations can be achieved at a given time. This problem was dealt with in [5] for systems of spin $\frac{1}{2}$ particles and the research there was motivated by the problem of controlling the state of quantum bits in order to achieve prescribed logic operations in quantum computing [6]. For any system of this kind the describing model is given by Schrödinger equation with the control multiplying the state variable. This is a right-invariant bilinear system whose state varies on a compact Lie group.

Controllability of systems on Lie groups was dealt with in the classical paper [7] and in a number of following papers. The survey article [8] and the book [9] give an up-to-date account of the main results and we refer to them for further references on this topic. It is a problem of great fundamental and practical importance to characterize the set of states that can be obtained in arbitrary small time. This set is in general not the whole Lie group even if the control is allowed to be a general Lebesgue measurable function (see Example 8.1 in [7]). To the best of the author's knowledge, a systematic study of this set has not been carried out in the literature, although sufficient conditions are available for it to be the whole Lie group [10]. In the present paper, we present a study of the set of states reachable from the identity of the group at any arbitrary time. We prove that this set is either empty or it is dense in a Lie subgroup of the underlying Lie group. If an additional regularity assumption is verified (Small time local controllability of the identity of the group) then this set is, in fact, a connected Lie subgroup of the underlying group. The paper is organized as follows. Sections II through IV are of general interest since they deal with controllability properties of general right invariant bilinear systems on compact Lie groups. In particular, in Section II we describe the mathematical model we want to study and give the basic definitions. We also prove a sufficient condition for the set of states reachable at any arbitrary time from the identity to be empty. This motivates the study of this set in the following two sections. In Section III, we prove that this set is either empty or is dense in a Lie subgroup of the underlying Lie group and in Section IV we relate the property of this set to be a connected Lie subgroup to the small time local controllability of the identity of the group. If this is verified, given the correspondence between connected Lie subgroups and Lie subalgebras, the problem of characterizing the set of states reachable from the identity at any arbitrary time can be approached studying the structure of the Lie algebra. Sections V and VI contain application of these results to the system of a particle with spin in an electro-magnetic field. In particular, in Section V the model for this system is described as a right invariant bilinear system on a compact Lie group and the controllability analysis for this system is presented in Section VI.

II. Systems on compact Lie groups

In this section we study general systems of the form

$$\dot{X} = AX + \sum_{i=1}^{m} B_i X u_i. \tag{1}$$

The state X varies on a compact matrix Lie group G while the matrices A, B_i, $i = 1, ..., m$, are constant matrices belonging to the corresponding Lie algebra $\mathcal{G}$. The restriction to *matrix* Lie groups is not necessary in the following and is considered here only for sake of concreteness. The control functions u_i, $i = 1, ..., m$, are piecewise continuous functions defined on some interval of $\mathbf{R}^+$. The matrices B_i, $i = 1, ..., m$, are assumed to be linearly independent although this is done without loss of generality. In fact, it is always possible to reduce the analysis of the behavior of the system (1) to this case by opportunely redefining the control functions. System (1) is right invariant in that if $X(t)$ is a solution corresponding to the initial condition equal to the identity matrix, the solution corresponding to the initial condition F is given by $X(t)F$.

The following Lie algebras and Lie groups are associated to the system in (1)

- $\mathcal{L}$ is the Lie algebra generated by $\{A, B_1, ..., B_m\}$ and $e^{\mathcal{L}}$ is the corresponding connected Lie subgroup of G.
- $\mathcal{L}_0$ is the ideal in $\mathcal{L}$ generated by $\{B_1, ..., B_m\}$ and $e^{\mathcal{L}_0}$ is the corresponding connected Lie subgroup of G.
- $\mathcal{B}$ is the Lie algebra generated by $\{B_1, ..., B_m\}$ and $e^{\mathcal{B}}$ is the corresponding connected Lie subgroup of G.

Notice that $\mathcal{L}_0$ has co-dimension 0 or 1 in $\mathcal{L}$ according to whether A is or is not in $\mathcal{L}_0$.

The following sets of states reachable from the identity I are associated to the system (1).

- $R(T)$; The set of all the possible values for $X(T)$ (solution of (1) at time T with initial condition equal to the identity I) obtained by varying the controls $u_1, ..., u_m$, in the set of all the piecewise continuous functions defined in $[0, T]$. This is also expressed by saying that, for every $X_f \in R(T)$, there exists a piecewise continuous control function defined in $[0, T]$ which *drives* the state of the system from the identity to X_f.
- $\mathcal{R}(\leq T) := \cup_{0 \leq t \leq T} R(t)$.
- $\mathcal{R} := \cup_{0 \leq t < \infty} R(t)$.

We have $R(0) = \mathcal{R}(\leq 0) = \{I\}$, and, by right invariance, the set of states reachable *from* a point $X \in G$ are given by $R(T)X$, $\mathcal{R}(\leq \mathcal{T})X$ and $\mathcal{R}X$, respectively. Therefore a study of the states reachable from the identity gives information on the states reachable from any other point. It will be useful sometimes to consider the sets $R^{-1}(T)$. These are the sets of all the matrices X in G for which there exist control functions, $u_1, ..., u_m$, driving the solution of (1) from $X \in R^{-1}(T)$ to the identity I, in time T. This is sometimes called the set of states *controllable* to the identity. By the right invariance property, the set of all the states that can be driven to a state X in time T is given by $R^{-1}(T)X$. It also follows from the right invariance that $R^{-1}(T) = [R(T)]^{-1}$ and that if X is an interior point of $R(T)$ then X^{-1} is an interior point of $R^{-1}(T)$.

From the results of [7], we have that $\mathcal{R} = G$ if and only if $\mathcal{L} = \mathcal{G}$ and, more in general, $\mathcal{R} = e^{\mathcal{L}}$. Moreover there exists a time T such that $\mathcal{R}(\leq T) = e^{\mathcal{L}}$. At every time t, $R(t) \subseteq e^{At}e^{\mathcal{L}_0}$, and the interior of $R(t)$ with respect to the topology of $e^{At}e^{\mathcal{L}_0}$ is not empty. It also follows from a result in [10] that, if $\mathcal{B} = \mathcal{L}$, then $R(t) = e^{\mathcal{L}}$, for every $t > 0$. This is the case for homogeneous systems ($A = 0$) [11].

The main topic of the following two sections is the study of the set of states reachable at any arbitrary time in the cases where the above recalled condition of [10] does not guarantee that it is equal to $e^{\mathcal{L}}$. More specifically, we are interested in the study of the set

$$\mathcal{A} := \cap_{t>0} R(t). \tag{2}$$

The examples in [5] [7] show that the set $\mathcal{A}$ might not be the whole $e^{\mathcal{L}}$. In fact, it may be empty, as the following Proposition shows.

Proposition 2.1. *If $\mathcal{L}_0$ has co-dimension 1 in $\mathcal{L}$ then $\mathcal{A}$ is empty.*

Proof. It follows from the above recalled result in [7] that, for every $t > 0$,

$$R(t) \subseteq e^{At}e^{\mathcal{L}_0}, \tag{3}$$

and therefore $\mathcal{A} \subseteq \cap_{t>0} e^{At}e^{\mathcal{L}_0}$. The right hand side of this inclusion is the empty set, if $A \notin \mathcal{L}_0$. In order to see this, assume $\cap_{t>0} e^{At}e^{\mathcal{L}_0} \neq \emptyset$ and pick $\tau_1, \tau_2 > 0$, with $\tau_2 - \tau_1 = t$, such that

$$e^{A\tau_1}F_1 = e^{A\tau_2}F_2, \tag{4}$$

for some F_1 and F_2 in $e^{\mathcal{L}_0}$. From this, it follows that

$$e^{A\tau_1}F_1F_2^{-1}e^{-A\tau_1} = e^{At} \in e^{\mathcal{L}_0}, \tag{5}$$

since $e^{\mathcal{L}_0}$ is a normal subgroup ([12] pg. 106 ff.). The fact that $e^{At} \in e^{\mathcal{L}_0}$, for all $t \in \mathbf{R}$ implies $A \in \mathcal{L}_0$, which we have excluded. □

In the following, we consider the system (1) as varying on $e^{\mathcal{L}}$ and the topology on $e^{\mathcal{L}}$ is the one induced by the one of G. Since we will be studying the set $\mathcal{A}$, we can assume, from the previous proposition, that $\mathcal{L}_0 = \mathcal{L}$. Since the interior of $R(t)$ is not empty in $e^{At}e^{\mathcal{L}_0}$ [7], for every t, we have, in our case, that $R(t)$ has nonempty interior in $e^{\mathcal{L}}$ for every t.

III. Set of states reachable at any arbitrary time

In the following three theorems, we assume that $\mathcal{A}$ is not empty. This can be checked, for example, by constructing a class of controls (for example constant controls) steering to a fixed point in G (for example the identity) in arbitrary time.

Theorem 3.1. *Assume $\mathcal{A}$ is not empty, then it is a semigroup and $\bar{\mathcal{A}}$ is a Lie subgroup of G, in particular it contains the identity I.*

Proof. If $X_1 \in R(t_1)$, and $X_2 \in R(t_2)$, then, by right invariance $X_2 X_1 \in R(t_1 + t_2)$. If X_1 and X_2 are in $\mathcal{A}$, t_1 and t_2 can be chosen positive but otherwise arbitrary, therefore $t_2 + t_1$ is also arbitrary and $X_2 X_1 \in \mathcal{A}$. This shows that $\mathcal{A}$ is a semigroup. Since $\mathcal{A}$ is a semigroup, so is $\bar{\mathcal{A}}$. The previous argument also shows that if X is in $\mathcal{A}$ then X^n is in $\mathcal{A}$ for every positive integer n. At this point, we follow an idea of [7] (Thm. 6.5) to prove that X^{-1} is in $\bar{\mathcal{A}}$. Because of compactness, the sequence $\{X^n\}$ has a convergent subsequence $X^{n(k)}$. The sequence of elements in $\mathcal{A}$, $\{X^{n(k+1)-n(k)-1}\}$, converges, as k tends to infinity, to X^{-1} and therefore $X^{-1} \in \bar{\mathcal{A}}$. Since $\bar{\mathcal{A}}$ is a closed subgroup of the Lie group G it is a Lie subgroup of G (see e.g. [13] pg. 110). $\square$

Theorem 3.2 *Assume $\mathcal{A}$ is not empty. If $t_1 < t_2$, then $\bar{R}(t_1) \subseteq \bar{R}(t_2)$.*

Proof. We prove $R(t_1) \subseteq \bar{R}(t_2)$. Let X be an element of $R(t_1)$ and $\{X_n\}$ a sequence of elements in $\mathcal{A}$ converging to the identity which is in $\bar{\mathcal{A}}$ by the previous Theorem. Since every X_n is in $R(t_2 - t_1)$, all the elements of the sequence $\{XX_n\}$ are in $R(t_2)$ and since this sequence converges to X, $X \in \bar{R}(t_2)$. This also proves that $\bar{R}(t_1) \subseteq \bar{R}(t_2)$. Notice that we also have $intR(t_1) \subseteq intR(t_2)$. This follows immediately from the general property (see e.g. [10]) $intR(t) = int\bar{R}(t)$, for every $t > 0$. $\square$

Theorem 3.3 *Assume $\mathcal{A}$ is not empty. Then, $e^{\mathcal{B}} \subseteq \cap_{t>0} \bar{R}(t)$.*

Proof. Assume $X_f \in e^{\mathcal{B}}$, then there exists an integer $l > 0$ and l real numbers $\alpha_1, ..., \alpha_l$ such that (see Lemma 6.2 in [7])

$$X_f = e^{\alpha_l B_{i_l}} e^{\alpha_{l-1} B_{i_{l-1}}} \cdots e^{\alpha_1 B_{i_1}}, \tag{6}$$

with $\{i_1, i_2, ..., i_l\} \in \{1, 2, ..., m\}$. Then the piecewise constant control defined on the interval $[\frac{j-1}{n}, \frac{j}{n})$, $j = 1, ..., l$, as

$$u_{i_j} = n\alpha_j, \qquad u_k = 0, \quad k \neq i_j, \tag{7}$$

gives for the state of system (1) at time $\frac{l}{n}$

$$X_n := e^{(A + n\alpha_l B_{i_l})\frac{1}{n}} e^{(A + n\alpha_{l-1} B_{i_{l-1}})\frac{1}{n}} \cdots e^{(A + n\alpha_1 B_{i_1})\frac{1}{n}}. \tag{8}$$

We have that $X_n \in R(\frac{l}{n})$ and

$$\lim_{n \to \infty} X_n = X_f. \tag{9}$$

Fix $t > 0$. For every $n > \frac{l}{t}$, using Theorem 3.2, we have

$$X_n \in \bar{R}(t). \tag{10}$$

This implies $X_f \in \bar{R}(t)$, and since t is arbitrary, we have $X_f \in \cap_{t>0}\bar{R}(t)$. □

IV. Consequences of Small Time Local Controllability

The three theorems proved in the previous section can all be sharpened if we assume *Small Time Local Controllability* for the identity element I of the group ($STLCI$). $STLCI$ means that there exists a time $T > 0$ such that the identity is in the interior of the reachable set $R(t)$ for every t, $0 < t \leq T$. This is readily seen to imply that the identity is in the interior of the reachable set $R(t)$ for every $t > 0$. This also implies that the identity is in the interior of $R^{-1}(t)$ and, by right invariance, that every point X is an interior point of the set of states controllable to X in time t, which is $R^{-1}(t)X$. The main result will be Theorem 4.3 which states that under this assumption the set $\mathcal{A}$ has the structure of a Lie group whose subalgebra contains $\mathcal{B}$. The following two lemmas contain generalizations of Theorems 3.1 through 3.3 when $STLCI$ is verified.

Lemma 4.1. *$STLCI$ implies that $\mathcal{A}$ is not empty and it is a closed Lie subgroup of G*

Proof. It is obvious that $\mathcal{A}$ is not empty since it contains at least the identity. From Theorem 4.1, all we have to prove is that $\mathcal{A}$ is closed. To see this notice that if $I \in intR(t)$ then $I \in int[R(t)]^{-1}$, namely I is in the interior of the set of states controllable to the identity in time t. This means, by right invariance, that every $X \in e^{\mathcal{L}}$ is in the interior of the set of states controllable to X in time t, which is $[intR(t)]^{-1}X$. Now, pick $X \in \bar{\mathcal{A}}$. There exists an element $Y \in \mathcal{A} \cap [intR(t)]^{-1}X$. The point Y can be reached from I in time t and X can be reached from Y in time t. Therefore $X \in R(2t)$, and since t is arbitrary, $X \in \mathcal{A}$. □

Lemma 4.2. *Assume STLCI is verified. Then $t_1 < t_2$ implies that*

$$R(t_1) \subseteq R(t_2). \tag{11}$$

Moreover $\cap_{t>0}\bar{R}(t) = \cap_{t>0}R(t) := \mathcal{A}$. As a consequence, $e^{\mathcal{B}} \subseteq \mathcal{A}$.

Proof. The proof of (11) follows immediately from the fact that $I \in R(t)$ for every t. As for the second statement, assume that $X_f \in \bar{R}(\tau)$, $\forall \tau > 0$. Pick $\tau = t - \epsilon$, with $t > \epsilon > 0$. $STLCI$ implies that X_f is in $int(R^{-1}(\epsilon)X_f)$. From $X_f \in \bar{R}(t-\epsilon)$, we obtain that $R(t-\epsilon) \cap R^{-1}(\epsilon)X_f$ is not empty. If Y is a point in it, we can steer the state of the system from the identity to Y in time $t - \epsilon$ and from Y to X_f in time ϵ. Therefore $X_f \in R(t)$ and since t is arbitrary $X_f \in \mathcal{A}$. The fact that $e^{\mathcal{B}} \subseteq \mathcal{A}$ is immediate from Theorem 3.3. □

Theorem 4.3. *If STLCI is verified, then $\mathcal{A}$ is a closed, connected Lie subgroup of G whose Lie algebra contains $\mathcal{B}$.*

Proof. The result is already contained in Lemmas 4.1 and 4.2, except for the fact that $\mathcal{A}$ is connected. However, since $R(t)$ is path-connected, for every $t > 0$ (see [7] Lemma 4.4), so is $\bar{R}(t)$. From Lemma 4.2, $\mathcal{A}$ is the intersection of a decreasing sequence of continua (the compact and connected sets $\bar{R}(t)$) and therefore is itself a continuum (see [14] Theorem 7 pg. 212) and in particular it is connected. The second statement follows from the one to one correspondence between subalgebras of $\mathcal{G}$ and connected Lie subgroups of G. □

From Theorem 4.3, it follows that, once $STLCI$ is proved, one can approach the problem of characterizing $\mathcal{A}$ at the Lie algebra level. In fact $\mathcal{A}$ is a Lie group whose Lie algebra contains $\mathcal{B}$. One can consider all the Lie algebras containing $\mathcal{B}$. In some cases, as in the case of systems with spin angular momentum considered in the next section, the only Lie algebra containing $\mathcal{B}$ is $\mathcal{L}$, so that if one proves that not every state can be reached in arbitrary time then it immediately follows that $\mathcal{A} = e^{\mathcal{B}}$.

There have been many studies concerning the property of Small Time Local Controllability for a given point in the state space of a nonlinear system. Many results (see e.g. [8]) deal with the case in which the point is an equilibrium point of the system when the control is set to zero. We give here a criterion, based on the Maximum Principle [15] [8]. The proof is a generalization of the one used in [5] for the case of two-level quantum systems. We use the following notation: $ad_X^0 Y := Y, ad_X^k Y = [X, ad_X^{k-1} Y]$.

Theorem 4.4. *Assume there exists a time T such that, for every $\tau \leq T$, there exists a piecewise constant control u_τ steering from the identity to the identity in time τ. Denote the values assumed by the function u_τ by $\mathcal{U}_\tau := \{u_1, u_2, ..., u_{k(\tau)}\}$. For a value u_j define the matrix*

$$F_j := A + \sum_{i=1}^{m} B_i u_{ij}, \tag{12}$$

where u_{ij}, $i = 1, ..., m$, are the components of u_j. Assume that for every τ, there exists a $u_j \in \mathcal{U}_\tau$ such that

$$ad_{F_j}^n B_i, \qquad n = 0, 1, 2, ..., k, \qquad i = 1, ..., m, \tag{13}$$

span the whole Lie Algebra $\mathcal{L}$. Here k is the dimension of the Lie group. Then, the system has the $STLCI$ property.

Proof. We apply the Maximum Principle for systems on Lie groups [15] in the form that gives information on the structure of reachable sets [8]. We obtain that the identity is in the interior of the reachable set at time τ, $R(\tau)$, if the only matrix $M \in \mathcal{L}$ such that

$$< M, X^{-1}(t)(A + \sum_{i=1}^{m} B_i u_{ij}) X(t) >= \min_{v_1,...,v_m} < M, X^{-1}(t)(A + \sum_{i=1}^{m} B_i v_i) X(t) >= constant, \tag{14}$$

is the zero matrix [15]. Since u_j is finite, (14) implies that

$$< M, X^{-1}(t) B_i X(t) >\equiv 0, \qquad i = 1, ..., m, \tag{15}$$

in some interval of positive measure, $[t_1, t_2] \subseteq [0, \tau]$. Differentiating (15) n times at t_1, one obtains

$$< M, X^{-1}(t_1) ad_{F_j}^n B_i X(t_1) >= 0, \qquad n = 1, 2, ..., \tag{16}$$

and since $ad_{F_j}^n B_i$, $n = 0, 1, ..., k$, $i = 1, 2, ..., m$ span the whole Lie algebra $\mathcal{L}$, (16) and (15) with $t = t_1$ imply $M = 0$. $\square$

V. Particles with spin in an electro-magnetic field

In this and the following section we apply the results obtained in the previous sections and perform the controllability analysis of a class of quantum systems. We consider a particle with spin and all the other degrees of freedom ignored under the action of an externally applied electro-magnetic field. We review the basic facts about the mathematical model in this section (see e.g. [16]) and perform the controllability analysis in the next section.

The (time varying) Hamiltonian describing the system is given by

$$H(t) := \gamma \mathbf{J} \cdot \mathbf{B} := \gamma (J_x B_x(t) + J_y B_y(t) + J_z B_z(t)). \tag{17}$$

In (17) γ is the *gyromagnetic ratio* of the particle, $J_{x,y,z}$ are the x, y, z components of the spin angular momentum operators and $B_{x,y,z}$ are the (time varying) components of the electro-magnetic field which play the role of control. $J_{x,y,z}$ are Hermitian operators on the underlying Hilbert space which satisfy the *fundamental commutation relations*

$$[J_x, J_y] = i\hbar J_z, \qquad [J_y, J_z] = i\hbar J_x, \qquad [J_z, J_x] = i\hbar J_y. \tag{18}$$

The theory of angular momentum in quantum mechanics originates from these relations (see e.g. [16] Chpt. 3). The evolution (rotation) operator X is obtained by solving Schrödinger equation

$$i\hbar \dot{X}(t) = H(t) X(t), \tag{19}$$

with initial condition $X(0)$ given by the identity operator. The Hamiltonian is given in (17), and we are interested here in a controllability analysis of this system, namely we want to investigate what are the rotations that can be achieved in a particular configuration for system (19).

The spin of a particle may assume a value j which is either a positive integer or a positive half integer. For a particle with spin j the operators J_x,, J_y, J_z can be represented by $2j+1 \times 2j+1$ Hermitian matrices which we still denote by J_x, J_y, J_z. Defining $S_{x,y,z} := \frac{-iJ_{x,y,z}}{\hbar}$, we can write Schrödinger equation (19), (17) for the evolution matrix as

$$\dot{X}(t) = \gamma (S_x B_x(t) + S_y B_y(t) + S_z B_z(t)) X(t), \tag{20}$$

which has to be solved with $X(0) = I_{2j+1 \times 2j+1}$. The matrices S_x, S_y, S_z satisfy the commutation relations corresponding to (18)

$$[S_x, S_y] = S_z, \qquad [S_y, S_z] = S_x, \qquad [S_z, S_x] = S_y. \tag{21}$$

They are skew-Hermitian and it follows immediately from (21) that they have zero trace. Therefore, they span a three-dimensional subalgebra of the Lie algebra $su(2j+1)$ of skew-Hermitian $2j+1 \times 2j+1$ matrices with zero trace. We denote this 3-dimensional Lie algebra by $\mathcal{G}_j$ and the corresponding connected Lie subgroup of $SU(2j+1)$ by G_j. An inner product $< \cdot, \cdot >$ can be defined in $\mathcal{G}_j$ by

$$< A, B > := Trace(AB^*), \tag{22}$$

where B^* denotes the conjugate transpose of the matrix B. The Lie algebra $\mathcal{G}_j$ is semisimple (it is not Abelian and it has no Abelian ideal) and the Lie subgroup G_j is compact. The first statement can be verified by computing the Killing matrix $K_{ik} = Trace(AdS_i AdS_k)$, $i, k = x, y, z$ and verifying that it is not degenerate (namely its determinant is different from zero). AdS_i is the matrix representation of the linear operator acting on $\mathcal{G}_j$ by $Y \rightarrow [S_i, Y]$. This is Cartan's criterion of semisimplicity (see e.g [17] pg. 14). Compactness can be checked by applying Weyl's theorem (see e.g. [17] pg. 20) checking that the Killing matrix K_{ik} is negative definite. Both these results can also be obtained by noticing that for every j, the Lie algebra $\mathcal{G}_j$ is isomorphic to the Lie algebra $su(2)$ (or $so(3)$), of skew-Hermitian 2×2 matrices with zero trace (antisymmetric real 3×3 matrices) and therefore the corresponding Lie group is isomorphic either to the Lie group of 2×2 special unitary matrices $SU(2)$ or to the Lie group of 3×3 special orthogonal matrices $SO(3)$, and therefore semisimplicity and compactness follow from known properties of the Lie algebras $su(2)$ and $so(3)$ and the corresponding Lie groups $SU(2)$, $SO(3)$.

The above recalled result about the isomorphism between the Lie group G_j and $SU(2)$ or $SO(3)$ is crucial to the controllability analysis for spin angular momentum systems that will follow because it reduces the study to two cases: the Lie groups $SU(2)$ and $SO(3)$. This result appeared in a study by E. P. Wigner ([18] pp. 163-168). We state it in the following theorem.

Theorem 5.1 *G_j is isomorphic to $SO(3)$ for j integer and isomorphic to $SU(2)$ for j half-integer.*

VI. Controllability of Spin Angular Momentum

We refer to the system in the general form (1) that we repeat here

$$\dot{X} = AX + \sum_{i=1}^{m} B_i X u_i, \tag{23}$$

where it is now understood that the matrices $A, B_1, ..., B_m$ are in the Lie algebras $\mathcal{G}_j$, as defined in the previous Section. This general form include all the possible geometric configurations that can be realized in a laboratory. For example, one could apply a constant electro-magnetic field and a time varying one at an angle of 30^o in the $x-y$ plane so that both the components of the field in the x and y directions have a constant component (modeled by the matrix A) and a time varying component.

First notice that if two (or more) inputs are available, since we have assumed $B_1, ..., B_m$ linearly independent, $B_1, ...B_m$ generate the whole Lie algebra $\mathcal{G}_j$ [19]. In this case, one can apply a result in [10] (Theorem 5.3) to conclude that $R(t) = G_j$ for every t. Therefore the only nontrivial case is the single-input one. We also assume that A and B_1 are linearly independent in this case which, in physical terms, means that there are at least two non parallel directions for the driving electro-magnetic field. If this is not the case then the solution of (23) is just $X(t) = e^{\int_0^t A+B_1u(\tau)d\tau}$.

Consider now system (23) with a single input assuming the matrices A and B in $su(2)$ (or $so(3)$) and the corresponding solution X in $SU(2)$ (or $SO(3)$). This is always possible because of the Lie group isomorphism of Theorem 5.1. Explicit expressions for this isomorphism are given for example in [20] (pp. 135-141). We write the system as

$$\dot{X} = AX + BXu. \tag{24}$$

Consider the constant input $u = -\frac{<A,B>}{<B,B>} + v$. The eigenvalues of the matrix $A + Bu$ are $0, \pm i\sqrt{v^2 + p^2}$ in the $so(3)$ case and $\pm i\sqrt{\frac{<B,B>}{2}v^2 + p^2}$, in the $su(2)$ case. Here p is the magnitude of the purely imaginary conjugate eigenvalues of $A - \frac{<A,B>}{<B,B>}B$. These expressions show that the nonzero eigenvalues can be made arbitrarily large in magnitude by choosing v large, and therefore the corresponding solution of (24) returns to the identity in arbitrary small time. This shows that the identity is in $R(t)$ for each t. Define $F := A - \frac{<A,B>}{<B,B>}B + Bv$. Since A and B are assumed to be linearly independent so are B and F. Recalling that $su(2)$ (and $so(3)$) have no two-dimensional subalgebras, it is easily seen that B, $ad_F B$ and $ad_F^2 B$ span the whole $su(2)$ (or $so(3)$) so that we can apply Theorem 4.4 to conclude that the identity is in the interior of the reachable set $R(t)$ for every t. Using again the fact that $su(2)$ ($so(3)$) does not have two dimensional subalgebras we conclude from Theorem 4.3 that the set of states reachable in arbitrary time $\mathcal{A}$ for this system is either the whole group or the subgroup $e^{\mathcal{B}}$, where, in this case, $\mathcal{B}$ is the one dimensional subalgebra generated by B in (24). However, the set of states reachable in arbitrary time is not the whole group. An example of this phenomenon was given in [7] for $SO(3)$ (Example 8.1 in [7]) and, in fact, this example is somehow canonical since *every* system on $SO(3)$ with one input has this property. In order to see this consider the system (24) and assume, without loss of generality, that A and B are orthogonal (we can always do this by shifting the input $u \to -\frac{<A,B>}{<B,B>} + v$). Moreover, by an appropriate change of coordinates and rescaling of time, we can always assume

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{25}$$

which is the system considered in [7]. The same steps can be carried on for systems varying on $SU(2)$ with the equation (24) and some algebraic manipulations show that not all the states can be obtained in arbitrary small time [5]. Alternatively one can use the two-to-one correspondence between $SU(2)$ and $SO(3)$ [5] [20] (pg. 124) and argue that, if all the states could be obtained in $SU(2)$, in arbitrary time, the same would be true for $SO(3)$. Applying the isomorphism of Theorem 5.1, We conclude with the following Theorem.

Theorem 6.1 *Consider a system with spin under the action of an electro-magnetic field as described by equation (24). Then the set of rotations (states) that can be obtained in arbitrary time is given by the one dimensional Lie subgroup corresponding to the one dimensional Lie algebra generated by the matrix B*

During the revision of the present paper the author learned about a recent paper [23] where the model of spin systems is considered in the context of the theory of symmetric spaces. The result of Theorem 6.1 can also be obtained applying the results in [23]. This can be done by noticing the isomorphism of $\mathcal{G}_j$ with $su(2)$ and the fact that system (24) (with A and B orthogonal) is underlying a Cartan decomposition [17] of the Lie algebra $su(2)$. If t_F denotes the infimum of the times $t \geq 0$ such that $X_f \in R(t)$ the only states X_f having $t_F = 0$ are the ones in $e^{\mathcal{B}}$. Using the fact that $I \in R(t)$, $\forall t > 0$, one can conclude that $e^{\mathcal{B}} = \mathcal{A}$ in this case. The analysis based on the results of [23] has the merit that can be easily generalized, to characterize, for a given time $T > 0$, the set of states that can be reached in time $T + \Delta$, $\forall \Delta > 0$. On the other hand, the derivation presented here, based on the general properties of the set $\mathcal{A}$, proved in the previous sections, gives further information concerning the $STLCI$ property of the model.

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[19] This claim is obvious if $m = 3$. If $m = 2$, assume $[B_1, B_2] = aB_1 + bB_2$ for some real coefficients a, b. Write B_2 as $B_2 = cB_1 + dF$ with F orthogonal to B_1. We get $[B_1, B_2] = (a + bc)B_1 + bdF$. Taking the inner product (22) by B_1 and F, and noticing that $[B_1, B_2]$ is perpendicular to both B_1 and F, we obtain $a+bc = 0$ and $bd = 0$, which implies $a = 0$ and $b = 0$, since d is different from zero.

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