a) Assume $X_2$ is a solution i.e.
\[ A X_1 = 0 \quad (1) \]
and $X_2$ is also a solution therefore
\[ A X_2 = C \quad (2) \]

Then we have
\[ A (X_1 + X_2) = A X_1 + A X_2 \quad (3) \]
from the distributive property of matrix multiplication. Using (1) and (2) in (3) we have
\[ A (X_1 + X_2) = 0 + 0 = 0 \]
which proves that \( X_1 + X_2 \) is a solution.

To prove that \( cX_1 \) is a solution, calculate

\[
A(cX_1) = cAX_1 = cO = 0
\]

where we used elementary properties of matrix multiplication and (1).

b) Consider the system

\[
X_1 + X_2 = 1
\]

The set of solutions is given by

\[
X_2 = \lambda \quad X_1 = 1 - \lambda
\]

as \( \lambda \) varies in \( \mathbb{R} \).

\[

\]
Two possible solutions are 
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = X_1 \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = X_2
\]

\[
X_1 + X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]
is not a solution as

\[1 + 1 = 2 \neq 1\]

Moreover, \[3X_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}\] is also not a solution as

\[3 + 0 = 3 \neq 1\]
c) Assume $X_1$ is a solution of $AX = B$ namely

$$AX_1 = B,$$ \hspace{1cm} (1)

and $X_2$ is a solution of $AX = 0$ i.e.

$$AX_2 = 0.$$ \hspace{1cm} (2)

Then, we have

(3) $A(X_1 + X_2) = AX_1 + AX_2 = B + 0 = B$

where we used elementary properties of matrix multiplication and (1) and (2).

Formula (3) shows that $X_1 + X_2$ is a solution of $AX = B$.
d) Assume that $B \neq 0$ and $AX = B$ has only one solution. Call this solution $X_1$, so that

$$AX_1 = B.$$ 

We want to show that $AX = 0$ has only the trivial solution $X = 0$. We proceed by contradiction. Assume $X_2 \neq 0$ is a nontrivial solution of $AX = 0$. Then from part c) we have that $X_1 + X_2$ is also a solution of $AX = B$. However, this contradicts the assumption that $X_1$ is the only solution of $AX = B$ because $X_1 + X_2 \neq X_1$ since $X_2 \neq 0$. 

a) Recall from Theorem 2.4 that for every matrix $A$, there exists a unique reduced row echelon form matrix $T$ which is not equivalent to $A$, namely

$$R A = T \quad (1)$$

where $R$ is the product of row operations.

From (1) we have:

$$R R^{-1} R A = T \quad (2)$$

for any row operation $R$ and since $R R^{-1}$ is the product of three row operations (2) shows that $R A$ is not equivalent to $T$. 
as well. Since $A$ and $RA$ are row equivalent to the same reduced row echelon form matrix $T$, they have the same rank.

(b) Assume $k$ of the $m$ rows of $A$ are zero and let $R$ the row operation which exchanges the row so that the $k$ zero rows are the last one. From part (a) we know:

1. \( \text{rank}(RA) = \text{rank}(A) \)

Let $P$ a row operation of the form

\[
P = \begin{bmatrix} R_1 & \mathbf{0} \\ \mathbf{0} & I_{n \times n} \end{bmatrix}
\]
where $R_1$ is the row operation which puts in reduced row echelon form the $m-k$ rows of $RA$. We have that $P(RA)$ is in reduced row echelon form and its last $n$ rows are zero so the maximum number of non-zero rows is $m-k$. This shows that

$\text{rank}(RA) \leq m-k$ and this along with (1) gives

$\text{rank}(A) \leq m-k$
c) Let $A$ be in r.r.e. form. Then the last $m - \text{rank}(A)$ rows of $A$ are zero, and therefore the last $m - \text{rank}(A)$ rows of $AB$ are also zero. Applying the result of b) with $AB$ replacing $A$, and noticing that $AB$ has the same number of rows as $A$ namely $m$, we have

$$\text{rank}(AB) \leq m - k = m - (m - \text{rank}(A)) = \text{rank}(A)$$

which proves the claim.
d) Let $T$ be the reduced row echelon form of $A$, namely,

$$RA = T$$

with $b$ in some operation. This gives

$$A = R^{-1} T \quad (1)$$

Now

$$\text{rank}(AB) = \text{rank}(R^{-1}TB) = \text{rank}(TB) \quad (2)$$

where in the last equality we used part a). However, using part c) we have

$$\text{rank}(TB) \leq \text{rank}(T) \quad (3)$$
Since from part a) and (1)

\[ \text{rank}(T) = \text{rank}(A) \] (4)

combining (2), (3), and (4), we obtain

\[ \text{rank}(AB) \leq \text{rank}(A) \]

as desired.