HWK # 3 Solutions

I

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a) Assume X_1 is a solution i.e.

$$AX_1 = 0 \qquad (1)$$

and X_2 is also a solution therefore

$$AX_2 = 0 \qquad (2)$$

Then we have

$$A(X_1 + X_2) = AX_1 + AX_2 \qquad (3)$$

from the distributive property of matrix multiplication. ~~Con~~ Using (1) and (2) in (3) we have

$$A(X_1 + X_2) = 0 + 0 = 0$$

which proves that $X_1 + X_2$ is a solution.

To prove that cX_1 is a solution,

Calculate

$$A(cX_1) = cAX_1 = c0 = 0$$

where we used elementary properties of matrix multiplication and (1).

b) Consider the system

$$x_1 + x_2 = 1$$

the set of solutions is given by:

$$x_2 = \lambda \qquad x_1 = 1 - \lambda$$

as λ varies in $\mathbb{R}$.

Two possible solutions are

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = X_1$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = X_2$

$$X_1 + X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is not a solution as

$$1 + 1 = 2 \neq 1$$

Moreover $3X_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is also not a solution as

$$3 + 0 = 3 \neq 1.$$

c) Assume X_1 is a solution of $AX = B$ namely

$$AX_1 = B, \quad (1)$$

and X_2 is a solution of $AX = 0$ i.e.

$$AX_2 = 0. \quad (2)$$

Then, we have

$$(3)\ A(X_1 + X_2) = AX_1 + AX_2 = B + 0 = B$$

where we used elementary properties of matrix multiplication and (1) and (2). Formula (3) shows that $X_1 + X_2$ is a solution of $AX = B$

d) Assume that $B \neq 0$ and $AX = B$ has only one solution. Call this solution X_1, so ~~that~~

$$AX_1 = B.$$

We want to show that $AX = 0$ has only the trivial solution $X = 0$. We proceed by contradiction. Assume $X_2 \neq 0$ is a nontrivial solution of $AX = 0$. Then from part c) we know that $X_1 + X_2$ is also a solution of $AX = B$. However this contradicts the assumption that X_1 is the only solution of $AX = B$ because $X_1 + X_2 \neq X_1$ since $X_2 \neq 0$.

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a) Recall from Theorem 2.4 that for every matrix A there exists a unique reduced row echelon form matrix T which is row equivalent to A, namely

$$\bar{R} A = T \qquad (1)$$

where $\bar{R}$ is the product of row operations. From (1) we have

$$\bar{R} R^{-1} R A = T \qquad (2)$$

for any row operation R and since $\bar{R} R^{-1}$ is the product of row operations (2) shows that RA is row equivalent to T

as well. Since A and RA are row equivalent to the same reduced row echelon form matrix T they have the same rank.

b) Assume k of the m rows of A are zero and let R the row operation which exchanges the row so that the k zero rows are the last one. From point a) we know:

(1) $\text{rank}(RA) = \text{rank}(A)$

Let P a row operation of the form

$$P = \left[\begin{array}{c|c} R_1 & 0 \\ \hline 0 & I_{n\times n} \end{array}\right]$$

where R_1 is the row operation which puts in reduced row echelon form the $m-k$ rows of RA. We have that

$P(RA)$ is in reduced row echelon form and its last k rows are zero so the maximum number of non zero rows is $m-k$. This shows that

$$\operatorname{rank}(RA) \leq m-k$$

and this along with (1) gives

$$\operatorname{rank}(A) \leq m-k$$

c) Let A be in r.r.e. form. Then the last $k \triangleq m - \operatorname{rank}(A)$ rows of A are zero and therefore the last ~~~~ k rows of AB are also zero. Applying the result of b) with AB replacing A and noticing that AB has the same number of rows as A namely m, we have

$$\operatorname{rank}(AB) \leq m - k = m - (m - \operatorname{rank}(A)) =$$
$$= \operatorname{rank}(A)$$

which proves the claim.

d) Let T be the reduced row echelon form of A, namely

$$RA = T$$

with R a row operation. This gives

$$A = R^{-1}T \quad (1)$$

Now

$$\operatorname{rank}(AB) = \operatorname{rank}(R^{-1}TB) = \operatorname{rank}(TB) \quad (2)$$

where in the last equality we used part a). Moreover using part c) we have

$$\operatorname{rank}(TB) \leq \operatorname{rank}(T) \quad (3)$$

Since (from part a) ~~rank~~ and (1)

$$\text{rank}(T) = \text{rank}(A) \quad (4)$$

combining (2) (3) and (4), we obtain

$$\text{rank}(AB) \leq \text{rank}(A)$$

as desired.