

# On the Observability and State Determination of Quantum Mechanical Systems

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## Abstract

We consider the problem of determining the state of a quantum system via a series of controlled evolutions and nonselective measurements of a given observable. Generalizing previous results to the case of possibly degenerate observables, we give a bound on the maximum number of parameters of the state that can be determined. We give a general algorithm for the determination of the maximum number of parameters. We also discuss how indirect measurement, when the system and probe are jointly controllable, allows one to determine all the parameters of the quantum state.

## 1 INTRODUCTION AND BACKGROUND

The introduction of control theory in quantum physics has provided new insight and powerful tools to drive the state of quantum systems. Several problems in areas such as molecular dynamics [10] and quantum information theory [8] can be seen as control problems and treated within the framework of quantum control. Recently, another important problem of quantum physics, the problem of determination of the quantum state, [3], [9], was studied from the viewpoint of control theory (see e.g. [4], [5]). In this context, the concept of *observability* plays a central role. Techniques of state determination in quantum physics go under the name of *quantum state tomography* (see e.g. [6] for a review). They are concerned with the determination of a quorum of observables from which the state of the system can be determined. From a control perspective, the

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combined effect of controlled dynamics and measurements has to be considered in the determination of the initial state.

In this paper, we consider the framework of [4] namely a finite dimensional quantum control system of the form

$$\dot{X}(t) = -iH(u(t))X(t), \quad X(0) = I_{n \times n}, \quad (1)$$

where  $X$  is the evolution operator and  $H$  is the Hamiltonian, depending on the control  $u$ . The state is described by a *density matrix*  $\rho$  which evolves as

$$\rho(t) = X(t)\rho(0)X^*(t), \quad (2)$$

where  $X(t)$  is the solution of (1). The dynamics in (1) is characterized by the *dynamical Lie algebra* defined as the Lie algebra  $\mathcal{L}$  generated by  $\text{span}_v\{-iH(v)\}$ . Every solution of (1) lies on the connected Lie group associated to the Lie algebra  $\mathcal{L}$ , which is denoted here by  $e^{\mathcal{L}}$ . It is assumed that we are able to measure the expectation value of a given observable  $S$ , namely

$$y := \text{Tr}(S\rho), \quad (3)$$

where  $S$  is an Hermitian matrix which can be assumed traceless without loss of generality. In this context, the system is said to be *observable* if there are no two indistinguishable states, namely two states that give the same output  $y$  for any control function  $u$ . The system is observable [4] if and only if

$$\text{span}_{X \in e^{\mathcal{L}}}\{iX^*SX\} = \mathfrak{su}(n), \quad (4)$$

or equivalently

$$\mathcal{V} := +_{k=0}^{\infty} \text{span}\{ad_{\mathcal{L}}^k iS\} = \mathfrak{su}(n). \quad (5)$$

The space  $\text{span}\{ad_{\mathcal{L}}^k iS\}$  is defined as spanned by the Lie brackets  $[R_1, [R_2, [\dots [R_k, iS]]]]$ , with  $R_j \in \mathcal{L}$ ,  $j = 1, \dots, k$ . More specifically, two states  $\rho_1$  and  $\rho_2$  are indistinguishable if and only if  $i(\rho_1 - \rho_2) \in \mathcal{V}^\perp$ , where  $\mathcal{V}^\perp$  is the orthogonal of  $\mathcal{V}$  in  $\mathfrak{su}(n)$ . The following fact holds [4].

**Proposition 1.1** Controllability i.e.  $\mathcal{L} = \mathfrak{su}(n)$  implies observability.

Upon measurement, the state of the system,  $\rho$ , is modified according to

$$\rho \rightarrow \Phi(\rho), \quad (6)$$

where the *operation*  $\Phi$  depends on the type of measurement performed (see e.g. [2]). A consequence of this state modification is that, even for observable systems, it might not be possible to determine all the  $n^2 - 1$  parameters of the initial density matrix by a sequence of measurements.

The contribution of this paper is organized as follows. In the next section, we consider the case of a sequence of Von Neumann measurements and provide a

bound on the number of independent parameters of the density matrix that can be obtained by a sequence of evolutions and measurements. The main technical part of the paper is presented in Section III, where we give an algorithm, namely a sequence of evolutions, which allows one to extract the maximum information on the initial state for a sequence of Von Neumann measurements. In Section IV, we argue that a sequence of *indirect* measurements will allow one to obtain *all* the parameters of the initial state. Concluding remarks are presented in Section V.

The problem treated in this paper was solved in [4] for the special case of nondegenerate observables. The algorithm we shall present here however *does not* reduce to the one of [4] in the nondegenerate case. The algorithm was applied in [1] to the specific example of two interacting spin  $\frac{1}{2}$  particles, subject to a measurement of the total magnetization.

## 2 SEQUENCES OF VON NEUMANN MEASUREMENTS

According to the Von Neumann measurement postulate (see e.g. [11]), immediately after the measurement of the expectation value  $Tr(S\rho)$ , the state  $\rho$  is modified (cf. (6)) according to Lüders formula

$$\rho \rightarrow \mathcal{P}(\rho) := \sum_{j=1}^r \Pi_j \rho \Pi_j, \quad (7)$$

where  $\Pi_j$  is the orthogonal projection onto the eigenspace of  $S$  corresponding to the eigenvalue  $\lambda_j$ ,  $j = 1, \dots, r$ . The multiplicity of the eigenvalue  $\lambda_j$  will be denoted by  $n_j$ .

We are concerned with experiments consisting of a sequence of evolutions  $X_1, \dots, X_k, \dots$  alternate to measurements. We study how the measurements can be used to determine the initial state  $\rho_0$ . The state, after the  $k$ -th measurement, is given by

$$\rho_k = \mathcal{P}(X_k(\cdots \mathcal{P}(X_3 \mathcal{P}(X_2 \mathcal{P}(X_1 \rho_0 X_1^*) X_2^*) X_3^*) \cdots) X_k^*), \quad (8)$$

while the result of the measurement is

$$y_k := Tr(S\rho_k) = \quad (9)$$

$$Tr(S(X_k(\cdots \mathcal{P}(X_3 \mathcal{P}(X_2 \mathcal{P}(X_1 \rho_0 X_1^*) X_2^*) X_3^*) \cdots) X_k^*)).$$

It was observed in [4] that, if the observable  $S$  is nondegenerate, then at most  $n - 1$  independent parameters of  $\rho_0$  can be determined with this method. Generalizing this, we can say that at most  $n_1^2 + n_2^2 + \cdots + n_r^2 - 1$  independent parameters can be obtained for a general observable with eigenvalues of multiplicity

$n_1, \dots, n_r$ . In order to see this, using elementary properties of the trace, we rewrite the expression of the measurement in (9) as

$$y_k := \text{Tr}(S\rho_k) = \quad (10)$$

$$\text{Tr}(X_1\rho_0X_1^*\mathcal{P}(X_2^*\cdots(X_{k-1}^*\mathcal{P}(X_k^*SX_k)X_{k-1}))\cdots X_2).$$

Assume, without loss of generality, that  $S$  is diagonal with equal eigenvalues arranged adjacent to each other in its diagonal. Every matrix of the type  $\mathcal{P}(\cdot)$  has the only elements possibly different from zero on the corresponding diagonal blocks. Therefore, it follows from (10) that only elements on the corresponding diagonal blocks of  $X_1\rho_0X_1^*$  affect the measurement result  $y_k$ . The number of these parameters is  $n_1^2 + n_2^2 + \cdots + n_r^2 - 1$ .

We observe that the best case is when  $S$  has only two eigenvalues and one of them has multiplicity 1 and the other one has multiplicity  $n - 1$ . In fact, it is easily proved that

$$\max_{r \geq 2} \max_{n_1 + n_2 + \cdots + n_r = n} n_1^2 + n_2^2 + \cdots + n_r^2 - 1 = (n - 1)^2. \quad (11)$$

In the following section, we give a general algorithm, namely a sequence of evolutions and measurements, from which it is possible to extract the *maximum number* of parameters.

### 3 AN ALGORITHM FOR THE DETERMINATION OF THE STATE

We shall assume, without loss of generality that the matrix  $S$  has zero trace. If this is not the case, we can always write  $S$  as  $S = \frac{\text{Tr}(S)}{n}I_{n \times n} + \tilde{S}$ , where  $I_{n \times n}$  is the identity and  $\tilde{S}$  has zero trace. The effect is a ‘shift’ in the values of the observations  $y_k$  in (10), which does not affect our considerations. It is also assumed, for obvious reasons, that  $S \neq 0$  and, without loss of generality, that  $S$  is diagonal and with equal eigenvalues arranged adjacent to each other and that the eigenvalue in the position (1,1),  $\lambda_1$ , is different from zero. Define

$$\tilde{\rho} := X_1\rho_0X_1^*. \quad (12)$$

The values of the observations  $y_k$  in (10) represent the components of  $\tilde{\rho}$  along  $S$ ,  $\mathcal{P}(X_2^*SX_2)$ ,  $\dots$ ,  $\mathcal{P}(X_2^*\cdots(X_{k-1}^*\mathcal{P}(X_k^*SX_k)X_{k-1}))\cdots X_2$ . Therefore our task is to find  $\bar{n} := n_1^2 + n_2^2 + \cdots + n_r^2 - 2$  unitary transformations  $X_2, \dots, X_{\bar{n}+1}$  such that the matrices  $S$ ,  $\mathcal{P}(X_2^*SX_2)$ ,  $\mathcal{P}(X_2^*\mathcal{P}(X_3SX_3)X_2)$ ,  $\dots$ ,  $\mathcal{P}(X_2^*\cdots(X_{\bar{n}}^*\mathcal{P}(X_{\bar{n}+1}^*SX_{\bar{n}+1})X_{\bar{n}}))\cdots X_2$  span all of  $\mathcal{P}(isu(n))$ . Here and in the following we denote by  $isu(n)$  the space of  $n \times n$  Hermitian matrices with zero trace.

### 3.1 Reduction of the problem

A task equivalent to the one above described is to find  $\bar{n}$  unitary matrices,  $X_2, \dots, X_{\bar{n}+1}$ , such that

$$\begin{aligned} \text{Tr}(S\Gamma) &= 0, \\ \text{Tr}(X_2^* S X_2 \Gamma) &= 0, \\ \text{Tr}(X_2^* \mathcal{P}(X_3^* S X_3) X_2 \Gamma) &= 0, \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned} \tag{13}$$

$$\text{Tr}(X_2^* \cdots (X_{\bar{n}}^* \mathcal{P}(X_{\bar{n}+1}^* S X_{\bar{n}+1}) X_{\bar{n}})) \cdots X_2 \Gamma = 0,$$

for  $\Gamma \in \mathcal{P}(su(n))$ , implies  $\Gamma = 0$ . Equations (13) can be rewritten as

$$\begin{aligned} \text{Tr}(S\Gamma) &= 0, \\ \text{Tr}(S X_2 \Gamma X_2^*) &= 0, \\ \text{Tr}(S X_3 \mathcal{P}(X_2 \Gamma X_2^*) X_3^*) &= 0, \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned} \tag{14}$$

$$\text{Tr}(S X_{\bar{n}+1} \cdots (X_3 \mathcal{P}(X_2 \Gamma X_2^*) X_3^*) \cdots X_{\bar{n}+1}^*) = 0.$$

We define

$$\Gamma_0 := \Gamma, \tag{15}$$

$$\Gamma_k := \mathcal{P}(X_{k+1} \Gamma_{k-1} X_{k+1}^*), \quad k = 1, \dots, \bar{n}, \tag{16}$$

so that the conditions (14) can be rewritten compactly as

$$\text{Tr}(S\Gamma_k) = 0, \quad k = 0, 1, 2, \dots, \bar{n}. \tag{17}$$

We need to exhibit an appropriate choice of transformations so that (17) implies  $\Gamma_0 = 0$ .

In the prescription of the transformations  $X_2, \dots, X_{\bar{n}+1}$ , we shall use planar rotations [7] of the form  $U_{k,l}$ , with  $k < l$ , defined as  $n \times n$  matrices equal to the identity except for the elements at the intersection between the  $k$ -th and  $l$ -th row and column which are occupied by

$$\bar{U}(\theta, \sigma) := \begin{pmatrix} \cos(\theta) & -\sin(\theta)e^{-i\sigma} \\ \sin(\theta)e^{i\sigma} & \cos(\theta) \end{pmatrix}. \tag{18}$$

### 3.2 Step 1 of the algorithm: Determination of the first $n - 2$ evolutions

The first  $n - 2$  evolutions of the algorithm are chosen according to the following prescription.

**Step 1:** We choose the first  $n - 2$  transformations (if  $n > 2$ ),  $X_2, \dots, X_{n-1}$ , as planar rotations

$$X_j := U_{k_j, l_j}(\frac{\pi}{4}, 0), \quad j = 2, \dots, n - 1, \quad (19)$$

such that the following conditions are satisfied.

1. For every  $j$ ,  $k_j$  and  $l_j$  refer to two different eigenvalues of  $S$ . Recall that  $S$  is diagonal and equal eigenvalues are arranged adjacent to each other. For each eigenvalue there is a set of indices corresponding to the diagonal elements occupied by that eigenvalue.
2. Every index in the set  $\{2, \dots, n\}$  appears at least once in the pairs  $(k_j, l_j)$ ,  $j = 2, \dots, n - 1$ .

**Example 3.1** A possible choice, according to the prescription of Step 1, for  $n = 5$ ,  $S = \text{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2)$ , is  $X_2 = U_{2,3}(\frac{\pi}{4}, 0)$ ,  $X_3 = U_{2,4}(\frac{\pi}{4}, 0)$ ,  $X_4 = U_{2,5}(\frac{\pi}{4}, 0)$ . Another choice is  $X_2 = U_{1,3}(\frac{\pi}{4}, 0)$ ,  $X_3 = U_{1,4}(\frac{\pi}{4}, 0)$ ,  $X_4 = U_{2,5}(\frac{\pi}{4}, 0)$ .

We have the following proposition.

**Proposition 3.2** With the choice of Step 1,

$$\text{Tr}(S\Gamma_k) = 0, \quad k = 0, \dots, n - 2, \quad (20)$$

(cf. (17)) implies

1. The diagonal elements of  $\Gamma_k$  are the same as the ones of  $\Gamma_0$ , for every  $k = 0, \dots, n - 2$ .
2. The diagonal elements of  $\Gamma_{n-2}$  (and therefore the ones of  $\Gamma_k$ ,  $k = 0, \dots, n - 3$ ) are all zero.
3.  $\Gamma_{n-2} = 0$  implies  $\Gamma_0 = 0$ .

*Proof.*

Consider any matrix  $P := \{p_{j,l}\} \in \mathcal{P}(\text{isu}(n))$ . A transformation  $P \rightarrow U_{j,l}(\frac{\pi}{4}, 0)PU_{j,l}^*(\frac{\pi}{4}, 0)$  only modifies the  $j$ -th and  $l$ -th rows and columns of  $P$ . In particular, both the  $(j, j)$  and the  $(l, l)$  entries become equal to  $\frac{p_{j,j} + p_{l,l}}{2}$ . The operation  $\mathcal{P}$  does not modify the diagonal entries. Now, applying this to any  $\Gamma_k$ ,  $k = 0, \dots, n - 3$ ,  $\text{Tr}(S\Gamma_k) = 0$  implies

$$\sum_{h=1}^n \lambda_h p_{h,h} = 0, \quad (21)$$

while  $Tr(S\Gamma_{k+1}) = 0$  implies

$$\sum_{h=1}^{j-1} \lambda_h p_{h,h} + \lambda_j \frac{p_{j,j} + p_{l,l}}{2} + \sum_{h=j+1}^{l-1} \lambda_h p_{h,h} + \quad (22)$$

$$\lambda_l \frac{p_{j,j} + p_{l,l}}{2} + \sum_{h=l+1}^n \lambda_h p_{h,h} = 0.$$

Subtracting (22) from (21), we obtain

$$(\lambda_j - \lambda_l)(p_{j,j} - p_{l,l}) = 0, \quad (23)$$

which, since  $\lambda_j \neq \lambda_l$ , implies  $p_{j,j} = p_{l,l}$ . Therefore, the  $j$ -th and  $l$ -th diagonal elements of  $\Gamma_k$  are equal and this implies that the  $j$ -th and  $l$ -th diagonal elements of  $\Gamma_{k+1}$ , which are given by  $\frac{p_{j,j} + p_{l,l}}{2} = p_{j,j} = p_{l,l}$ , are also equal and equal to the ones of  $\Gamma_k$ . Applying this to every  $k$ , we see that the diagonal entries are not changed in going from  $\Gamma_0$  to  $\Gamma_{n-2}$  and this proves claim 1) of the proposition. Because of the choice of  $X_2, \dots, X_{n-1}$ , this also shows that the last  $n-1$  diagonal elements of  $\Gamma_{n-2}$  are all equal. Therefore, from (20), calling  $p_{j,k}$  the entries of  $\Gamma_{n-2}$ , we obtain

$$\lambda_1 p_{1,1} + p_{2,2} \left( \sum_{h=2}^n \lambda_h \right) = 0. \quad (24)$$

Using the fact that  $S$  has zero trace, we have

$$\lambda_1(p_{1,1} - p_{2,2}) = 0, \quad (25)$$

which implies  $p_{1,1} = p_{2,2} = \dots = p_{n,n}$ , since we have assumed  $\lambda_1 \neq 0$ . Since we also have  $\sum_{h=1}^n p_{h,h} = 0$ , this also implies

$$p_{1,1} = p_{2,2} = \dots = p_{n,n} = 0, \quad (26)$$

which is claim 2) of the proposition.

In order to prove claim 3), consider again the transformation  $P \rightarrow U_{j,l}(\frac{\pi}{4}, 0) P U_{j,l}^*(\frac{\pi}{4}, 0)$  which only modifies the  $j$ -th and  $l$ -th rows and columns of  $P$ . We can also assume now that the diagonal elements of  $P$  as well as the ones of  $U_{j,l}(\frac{\pi}{4}, 0) P U_{j,l}^*(\frac{\pi}{4}, 0)$  are all zero and that  $P \in \mathcal{P}(su(n))$ . If  $p_{j(l)}$  is the  $j(l)$ -th row of  $P$  and  $p_{j(l)}^+$  the corresponding row after the transformation, it is a straightforward computation to check that

$$p_j^+ = \frac{1}{\sqrt{2}}(p_j - p_l), \quad p_l^+ = \frac{1}{\sqrt{2}}(p_j + p_l), \quad (27)$$

and the corresponding columns are obtained from the requirement that the matrix is skew-Hermitian. After applying  $\mathcal{P}$ , since  $j$  and  $l$  refer to two different eigenvalues, and denoting by  $p_{j(l)}^{++}$  the new rows, we have

$$p_j^{++} = \frac{1}{\sqrt{2}} p_j, \quad p_l^{++} = \frac{1}{\sqrt{2}} p_l. \quad (28)$$

Applying this in the transformation from  $\Gamma_k$  to  $\Gamma_{k+1}$ , we see that  $\Gamma_{k+1} = 0$  implies  $\Gamma_k = 0$ . Applying this several times, we obtain claim 3) of the proposition.  $\square$

The above proposition would already give a set of suitable transformations in the special case where  $S$  is nondegenerate and therefore  $\Gamma_0$  is diagonal. In the general case, it reduces the problem of finding  $\bar{n} - (n - 2) = n_1^2 + n_2^2 + \dots + n_r^2 - n$  transformations  $X_n, \dots, X_{\bar{n}+1}$ , so that

$$\text{Tr}(S\Gamma_k) = 0, \quad k = n - 2, n - 1, \dots, \bar{n}, \quad (29)$$

implies  $\Gamma_{n-2} = 0$ , when the diagonal elements of  $\Gamma_{n-2}$  are equal to zero.

### 3.3 Step 2 of the algorithm: Determination of the remaining $n_1^2 + n_2^2 + \dots + n_r^2 - n$ evolutions.

**Step 2** For every  $j := 1, \dots, r$ , choose  $n_j^2 - n_j$  transformations as follows:

For every  $f := 2, \dots, n_j$  choose  $2(f - 1)$  transformations in the order described in the following substeps f1 and f2 (we shall denote by  $X_{new}$  the new transformation to be added to the list and set  $\tilde{n}_j := n_1 + n_2 + \dots + n_{j-1}$  with  $\tilde{n}_1 = 0$ .)

**Substep f1**

$$X_{new} = U_{\tilde{n}_j+1, \tilde{n}_j+f}(\frac{\pi}{4}, 0) U_{\tilde{n}_j+1, l}(\frac{\pi}{4}, \frac{\pi}{2}), \quad (30)$$

and then

$$X_{new} = U_{\tilde{n}_j+1, \tilde{n}_j+f}(\frac{\pi}{4}, 0) U_{\tilde{n}_j+1, l}(\frac{\pi}{4}, 0), \quad (31)$$

where  $l$  is an arbitrary index corresponding to an eigenvalue different from  $\lambda_j$ .

**Substep f2**

If  $f > 2$ , for any  $h = 2, \dots, f - 1$ ,

$$X_{new} = U_{\tilde{n}_j+1, \tilde{n}_j+h}(\frac{\pi}{4}) U_{\tilde{n}_j+1, l}(\frac{\pi}{4}, \frac{\pi}{2}), \quad (32)$$

and then

$$X_{new} = U_{\tilde{n}_j+1, \tilde{n}_j+h}(\frac{\pi}{4}, 0) U_{\tilde{n}_j+1, l}(\frac{\pi}{4}, 0), \quad (33)$$

where  $l$  is an arbitrary index corresponding to an eigenvalue different from  $\lambda_j$ .

**Example 3.3** A choice of the remaining transformations for the case treated in Example 3.1, where  $n_1 = 2$  and  $n_2 = 3$  is as follows. Define (cf. (36), (37) below)

$$Y_j(k, t, l) := U_{\tilde{n}_j+k, \tilde{n}_j+t}(\frac{\pi}{4}, 0) U_{\tilde{n}_j+k, l}(\frac{\pi}{4}, \frac{\pi}{2}), \quad (34)$$

$$Z_j(k, t, l) := U_{\tilde{n}_j+k, \tilde{n}_j+t}(\frac{\pi}{4}, 0) U_{\tilde{n}_j+k, l}(\frac{\pi}{4}, 0). \quad (35)$$

Then, a possible choice, following Step 2, is  $X_5 = Y_1(1, 2, 3)$ ,  $X_6 = Z_1(1, 2, 3)$ ,  $X_7 = Y_2(3, 4, 1)$ ,  $X_8 = Z_2(3, 4, 1)$ ,  $X_9 = Y_2(3, 5, 1)$ ,  $X_{10} = Z_2(3, 5, 1)$ ,  $X_{11} = Y_2(3, 4, 1)$ ,  $X_{12} = Z_2(3, 4, 1)$ . As another possible choice, we can replace  $Z_2(3, 4, 1)$  with  $Z_2(3, 4, 2)$  for  $X_{12}$ .

The sequence in Step 2 is divided into subsequences, each corresponding to a block  $j$ . If  $\Gamma$  is a matrix of the sequence  $\{\Gamma_k\}$  at the beginning of the  $j$ -th subsequence and  $\Gamma^+$  the corresponding matrix at the end of the  $j$ -th subsequence, we have the following two properties:

**P-1** Formula (29) implies that the  $j$ -th blocks of  $\Gamma$  and  $\Gamma^+$  are zero.

**P-2** If the remaining blocks  $j + 1, \dots, r$  of  $\Gamma^+$  are zero, then so are the remaining blocks of  $\Gamma$ .

After the subsequence corresponding to the  $r$ -th block is completed, an induction argument, using **P-1** and **P-2** allows one to conclude  $\Gamma_{n-2} = 0$ , as desired. We prove properties **P-1**, **P-2** in the next proposition.

**Proposition 3.4** The subsequence of Step 2 corresponding to the  $j$ -th block satisfies properties **P-1** and **P-2**.

*Proof.* We have organized the sequence of transformations for the  $j$ -th block in subsequences each corresponding to a value of  $f$ ,  $f = 2, \dots, n_j$ . The subsequence corresponding to  $f$  contains  $2(f - 1)$  elements. Define

$$Y(1, t) := U_{\tilde{n}_j+1, \tilde{n}_j+t}(\frac{\pi}{4}, 0)U_{\tilde{n}_j+1, l}(\frac{\pi}{4}, \frac{\pi}{2}), \quad (36)$$

and

$$Z(1, t) := U_{\tilde{n}_j+1, \tilde{n}_j+t}(\frac{\pi}{4}, 0)U_{\tilde{n}_j+1, l}(\frac{\pi}{4}, 0), \quad (37)$$

where  $l$  is an arbitrary index corresponding to an eigenvalue different from  $\lambda_j$ . Then the subspace corresponding to  $f$  is given by

$$Y(1, f), Z(1, f), Y(1, 2), Z(1, 2), \quad (38)$$

$$Y(1, 3), Z(1, 3), \dots, Y(1, f - 1), Z(1, f - 1).$$

□

Let us study how a transformation  $Y(1, t)$  modifies a matrix  $\Gamma \in \mathcal{P}(su(n))$ , with zero diagonal. If  $l$  corresponds to a block  $\bar{\Gamma}_s$  in the expression  $\Gamma := \text{diag}(\bar{\Gamma}_1, \dots, \bar{\Gamma}_j, \dots, \bar{\Gamma}_s, \dots, \bar{\Gamma}_r)$ , then  $Y(1, t)$  only affects the  $j$ -th and  $s$ -th block rows and columns of  $\Gamma$ . In particular, the  $j$ -th diagonal block is the same as  $\bar{\Gamma}_j$  except for the following changes (the indices now refer to the submatrix  $\bar{\Gamma}_j$ , and we denote by  $\gamma_{zw}$  the corresponding  $zw$  entry):

1. The diagonal entries in the  $(1, 1)$  and  $(t, t)$  position (that were zero before the transformation) are now equal to  $\frac{\text{Im}(\gamma_{1t})}{2}$  and  $-\text{Im}(\gamma_{1t})$ , respectively.

2. The elements in the position  $(1, t)$  and  $(t, 1)$  are now both equal to  $\frac{Re(\gamma_{1t})}{\sqrt{2}}$ .
3. The remaining elements of the first row can be obtained from the formula

$$\gamma_{1z}^+ := \frac{\gamma_{1z} + i\gamma_{tz}}{2}, \quad z = 2, \dots, t-1, t+1, \dots, n_j, \quad (39)$$

while the remaining elements of the  $t$ -th row are obtained as

$$\gamma_{tz}^+ := \frac{i\gamma_{1z} + \gamma_{tz}}{\sqrt{2}}, \quad z = 2, \dots, t-1, t+1, \dots, n_j. \quad (40)$$

The elements of the first and  $t$ -th columns are also changed and they can be obtained from the fact that the block  $\bar{\Gamma}_j$  is skew-Hermitian.

The  $s$ -th diagonal block  $\bar{\Gamma}_s$  is also unchanged except for the following:

1. All the entries of the column and row corresponding to the index  $l$ , except the diagonal one, are to be multiplied by  $\frac{1}{\sqrt{2}}$ .
2. The diagonal entry in the position corresponding to the index  $l$  is given by  $\frac{Im(\gamma_{1t})}{2}$ .

If we apply these considerations to the transformation  $\Gamma_k \rightarrow \Gamma_{k+1}$ ,  $Tr(S\Gamma_{k+1}) = 0$  gives

$$(\lambda_s - \lambda_j)Im(\gamma_{1t}) = 0, \quad (41)$$

and since  $\lambda_s \neq \lambda_j$ , we have

$$Im(\gamma_{1t}) = 0. \quad (42)$$

From the above calculations, it also follows that, in the passage from  $\Gamma_k$  to  $\Gamma_{k+1}$ , 1)  $\Gamma_{k+1}$  has still zero diagonal if  $\Gamma_k$  does, 2) the  $j$ -th block of  $\Gamma_{k+1}$  is zero if and only if the same is true for the  $j$ -th block of  $\Gamma_k$  and 3) the blocks  $j+1, \dots, r$  of  $\bar{\Gamma}_{k+1}$  are zero if and only if so are the corresponding blocks of  $\Gamma_k$ .

By studying the transformation  $Z(1, t)$  one finds the same properties 1)-3) as above and conclude that the  $(1, t)$  and  $(t, 1)$  elements of the  $j$ -th block are equal to zero.

After the combined application of  $Y(1, t)$  and  $Z(1, t)$ , the  $j$ -th block of the current  $\Gamma$  matrix has the  $(1, 1)$ ,  $(1, t)$ ,  $(t, 1)$ ,  $(t, t)$  entries equal to zero, and the elements  $\gamma_{1z}^+$  of the first row are given by

$$\gamma_{1z}^+ := \left(\frac{1}{4} - \frac{i}{2\sqrt{2}}\right)\gamma_{1z} + \left(\frac{i}{4} - \frac{1}{2\sqrt{2}}\right)\gamma_{tz}, \quad (43)$$

$$z = 2, \dots, t-1, t+1, \dots, n_j,$$

and the elements of the  $t$ -th row are

$$\gamma_{tz}^+ := \left(\frac{1}{2\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\gamma_{1z} + \left(\frac{i}{2} + \frac{i}{2\sqrt{2}}\right)\gamma_{tz}, \quad (44)$$

$$z = 2, \dots, t-1, t+1, \dots, n_j.$$

Now consider the  $f$ -th subsequence of the subsequence for the  $j$ -th block. We shall prove that, after the  $f$ -th subsequence, the principal minor of order  $f$  of the  $j$ -th block is equal to zero. We proceed by induction on  $f$ . Before the  $f$ -th subsequence, by inductive assumption, the principal minor of order  $f$  has the form

$$F := \begin{pmatrix} 0_{f-1 \times f-1} & v \\ -v^* & 0 \end{pmatrix}, \quad (45)$$

for some  $f-1$ -dimensional vector  $v$ . This is true for  $f=2$  since the matrix we start with has zero diagonal. After  $Z(1, f)Y(1, f)$ , applying (43) (44), it has the form

$$F_0 := \begin{pmatrix} 0 & w & 0 \\ -w^* & 0_{f-2 \times f-2} & k\bar{w} \\ 0 & -k^*w & 0 \end{pmatrix}, \quad (46)$$

where  $w$  is a  $1 \times (f-2)$  row,  $k$  is a scalar different from zero. By induction on  $g$ , applying (43) and (44), after  $Y(1, f)$ ,  $Z(1, f)$ ,  $Y(1, 2)$ ,  $Z(1, 2)$ ,  $Y(1, 3)$ ,  $Z(1, 3), \dots, Y(1, g)$ ,  $Z(1, g)$ , in that order, one obtains a matrix of the form

$$F_g := \begin{pmatrix} 0_{g \times g} & M & 0_{g \times 1} \\ -M^* & 0_{(f-1-g) \times (f-1-g)} & m \\ 0_{1 \times g} & -m^* & 0 \end{pmatrix}, \quad (47)$$

where the  $g \times (f-1-g)$  matrix  $M$  has the form

$$M := \begin{pmatrix} w \\ k_2 w \\ k_3 w \\ \cdot \\ \cdot \\ k_g w \end{pmatrix}, \quad (48)$$

and the vector  $m$  is given by

$$m := k_1 w^* \quad (49)$$

for some  $(f-1-g)$ -dimensional row  $w$  and nonzero scalars  $k_1, \dots, k_g$ . When  $g = f-1$ , one obtains that the whole  $f \times f$  principal block is equal to zero. Applying this to  $f = n_j$  and using the general properties of the transformations  $Y, Z$  1)-3), one concludes that **P-1** and **P-2** are verified.  $\square$

## 4 INDIRECT MEASUREMENT

The limit on the number of parameters proved in Section 2 suggests that it may be possible to obtain all the parameters of the initial density matrix by

coupling the system with an auxiliary system, whose initial condition is known. Let us denote by  $\rho_S$  and  $\rho_P$  the initial conditions of the system and the auxiliary system (the ‘probe’), respectively, at the beginning of the measurement process. If  $n_P$  denotes the dimension of the probe,  $\rho_P$  contains  $n_P^2 - 1$  parameters, which are assumed known. If  $n_S$  denotes the dimension of the system,  $\rho_S$  contains  $n_S^2 - 1$  unknown parameters. The state of the total system, at the beginning of the measurement process, is given by

$$\rho_{TOT} = \rho_S \otimes \rho_P. \quad (50)$$

If we assume perfect coupling between the probe and the system, namely that the total system is controllable, we can obtain, as a result of the measurement procedure  $n_1^2 + n_2^2 + \dots + n_r^2 - 1$  parameters, where  $n_j$ ,  $j = 1, \dots, r$ , are the multiplicities of the eigenvalues of an observable  $S_{TOT}$  on the total system. We note here that if

$$n_1^2 + n_2^2 + \dots + n_r^2 \geq n_S^2, \quad (51)$$

it may be possible to adapt the algorithm in Section 3 so that exactly the *unknown* parameters, namely all the parameters of  $\rho_S$ , are determined from the measurements. We illustrate this here in the case where  $S_{TOT}$  is a measurement on the probe only, namely it is of the form

$$S_{TOT} := I_{n_S} \otimes S. \quad (52)$$

This is the case usually referred to as *indirect measurement* in the literature [2]. If  $S$  is diagonal,  $S_{TOT}$  is diagonal and, in appropriate coordinates, we can write it in block diagonal form, where every block is of the form  $\lambda I_{n_S}$ , with  $\lambda$  an eigenvalue of  $S$ . We can use  $X_1$  in (12) to modify the initial state  $\rho_{TOT}$  so that the unknown parameters of  $\rho_S$  appear in the blocks corresponding to the blocks of  $S_{TOT}$ . To simplify this task, we can first diagonalize  $\rho_P$  in (50) which is known and then use a transformation which permutes  $\rho_S$  and  $\rho_P$ . The resulting matrix has blocks on the diagonal which are proportional to  $\rho_S$ . Then another transformation is applied to make these blocks coincide with the ones of  $S_{TOT}$ . All the applied transformations do not depend on the value of  $\rho_S$  which is assumed unknown. At this point, we can apply the algorithm in this paper to determine *all* the values of the unknown parameters in  $\rho_S$ . We notice that, for indirect measurement of the form (52), any probe of dimension  $n_P \geq 2$ , independently of the initial condition, would allow to measure all of the unknown parameters of  $\rho_S$ . In fact, in this case, if  $\bar{n}_j$  are the multiplicities of the eigenvalues of  $S$  in (52), the number of unknown parameters determined through indirect measurement is  $n_S^2(\sum_j \bar{n}_j) - 1 > n_S^2 - 1$ .

## 5 CONCLUSIONS

Nonselective measurements of the expectation values of a given observable can be used, alternating with controlled evolutions, to determine the state of an

ensemble. In general only a limited number of independent parameters of the initial density matrix can be obtained. This number depends on the degree of degeneracy of the measured observable but it is never equal to the totality of the parameters of the density matrix.

In this paper, we have given a general algorithm, namely a sequence of controlled evolutions, which allows one to extract the maximum number of parameters of the density matrix for a given observable. We notice here that the minimum number of transformations needed was used in this algorithm. We also remark that, since a sequence of evolutions exists, it follows from our proof and an analyticity argument that almost every sequence of evolutions will give outputs that are linearly independent functions of the unknown parameters. Therefore almost any sequence would give an alternative algorithm. Our analysis only assumes controllability of the given system which implies observability. We have discussed how an indirect measurement where the total system (system+probe) is controllable, can be used, along with the algorithm presented here, to obtain complete information on the state.

The results of this paper can be reformulated in the language of *Positive Operator Valued Measures* (POVM). The algorithm in Section 3 can be seen as defining a POVM namely a set of positive operators with sum equal to the identity (see e.g [8]). In our case every operator corresponds to a sequence of possible outcomes of the selective measurements of the observable  $S$ . We have shown how, from the expectation values of these measurement results, it is possible to obtain data on the initial states. The same can be said for the indirect scheme of Section 4. In this case, the prescribed POVM is *informationally complete* in that the expectation values of the measurement results give complete knowledge of the initial state of the system.

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