A Short Introduction to Incompressibility Method Proofs for Random Graphs (CS633 Paper)

Chad Brewbaker

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Abstract

In this tutorial we will see how the Kolmogrov complexity of a string, the length of the shortest program that outputs that string, can be used to give us some property of binary strings of length $\binom{n}{2}$, and thus all graphs on n verticies.

1 Preliminary Definitions

Let $G = \{V, E\}$ be a simple graph on n verticies. The graph will be represented as its edge list, having $\binom{n}{2}$ bits, one to denote the presence or absence of each edge.

Pick your favorite universal Turing machine programming language:C, Java, Pascal, Scheme...

Let C(x) be the length of the shortest program that outputs the string x. We call this the Kolmogrov complexity of x.

Let K(x), C(x)≤K(x)≤C(x)+2log(C(x))+O(1), be the shortest program that outputs x and then halts. We call this the prefix Kolmogrov complexity of x.

Let C(x‖y) be the length of the smallest program that outputs x when given y as an input. We call this the conditional Kolmogrov complexity of x given y.

Let K(x‖y) be defined similarity, except the program halts after printing x. We call this the prefix conditional Kolmogrov complexity of x given y.

The randomness deficiency of a graph, $\delta(n)$ is defined as: $c(E(G)|n) \leq \binom{n}{2} - \delta(n)$, which is the number of bits we can compress the graph from its edge list representation. The notation $\delta(n)$ is taken from [5]. Try not to get it confused with the popular notation for the minimum vertex degree of a graph.

One last thing to remember. It takes $lg(n)$ bits to specify the number n, where $lg(n)$ is the base 2 logarithm.

2 Properties of random graphs from Li and Vitanyi

Here are some properties about random graphs as shown by Li and Vitanyi. They give a feel for how we can use compression to determine properties of random graphs.

(Property 1)[5]
A fraction of at least $1 - \frac{1}{2^{\delta(n)}}$ of all labeled graphs G on n verticies are $\delta(n)$ random. i.e can be compressed by δn bits.

(Property 2)[5]
All o(n)-random labeled graphs have $\frac{n}{4}$+o(n) disjoint paths of length 2 between each pair of verticies i,j. As a corollary all o(n)-random labeled graphs have diameter 2.

Diameter 2 means that the length of the maximum shortest path between any two verticies in the graph is 2.

A totally connected graph can be described by a O(1) length program, so we must look at diameters greater than or equal 2. If the diameter is three or greater, then two verticies in the graph have no common neighbor. Thus we can shorten the description by around 2*(n-2) bits for diameter 3, or around $2^{diameter()}$ in general. Thus, the large majority of graphs have diameter two.

(Property 3)[5]
If a graph G has randomness deficiency of O(log(n)) bits, then the largest clique in G has at most $\lfloor 2log(n) \rfloor + O(1)$ verticies.

The argument is that we can describe a size 2log(n) clique in O(log(n)) bits. Thus, any larger clique or anti-clique and we could compress the graph even more.

3 Following the pattern

We are starting to see a pattern with incompressibility method arguments:
1) Come up with a compression scheme given some graph invariant.
2) Solve for the compression equation:
(size of invariant compressed graph)≈ $\binom{n}{2}$

The following are some quick pen and paper calculations I did. Hopefully, they give you feel for the simplicity of incompressibility proofs.

3.1 Chromatic Number

First we will prove that the chromatic number of a random graph is around $\frac{n}{2lg(n)+1}$.
Incompressibility Argument:
A graph with chromatic number $\chi(G)$ can be represented in the following manner:

It takes $\chi(G) + \frac{n}{\chi(G)} * lg(n)$ to describe the color classes. Since verticies within the same color class have no edges between them we don't have to describe these edge relationships. Thus, only $\binom{n}{2} - \chi(G) * \binom{\frac{n}{\chi(G)}}{2}$ is required to list the edges between color classes. Thus, for random graphs:
$\chi(G) + \frac{n}{\chi(G)} * lg(n) + \binom{n}{2} - \chi(G) * \binom{\frac{n}{\chi(G)}}{2} \approx \binom{n}{2}$
$nlg(n) - \chi(G)\binom{\frac{n}{\chi(G)}}{2} \approx 0$
$nlgn \approx \chi(G)\frac{\frac{n}{\chi(G)}(\frac{n}{\chi(G)}-1)}{2}$
$2nlg(n) \approx \frac{n^2}{\chi(G)} - n$
$2lg(n) \approx \frac{n}{\chi(G)} - 1$
$2lg(n) + 1 \approx \frac{n}{\chi(G)}$
$\frac{1}{2lg(n)+1} \approx \frac{\chi(G)}{n}$
$\frac{n}{2lg(n)+1} \approx \chi(G)$
QED

3.2 The Maximum Clique and Maximum Independent(Stable) Set of Random Graphs

Since the maximum clique $\omega(G)$ is the same as the independence, or stability, number $\alpha(\bar{G})$ we expect these to be the same in a random graph. Given the maximum clique in a graph we can represent it in the following way:

$\omega(G) * lg(n)$ to describe verticies of the clique. For the rest of the edges we need $\binom{n}{2} - \binom{\omega G}{2}$. Thus an expected value of:
$\omega(G) * lg(n) + \binom{n}{2} - \binom{\omega G}{2} \approx \binom{n}{2}$
$\omega(G) * lg(n) - \binom{\omega G}{2} \approx 0$
$\omega(G) * lg(n) \approx \binom{\omega G}{2}$
$\omega(G) * lg(n) \approx \frac{\omega(G*(\omega(G)-1)}{2}$
$2\omega(G) * lg(n) \approx \omega(G)^2 - \omega(G)$
$2lg(n) \approx \omega(G) - 1$
$2lg(n) + 1 \approx \omega(G)$
QED

3.3 The Smallest Vertex Cover of Random Graphs

The smallest vertex cover of a graph is the smallest set of verticies such that every edge in the graph is adjacent to our set. We shall denote the vertex cover of a graph as $VC(G)$. Given the vertex cover of a graph we can represent it in the following way:

First describe the edges between the vertex cover verticies:$EC(G)lg(n)$. Then list the edges inside the cover $\binom{VC(G)}{2}$, then list edges between the cover and the other verticies $(n - VC(G))\binom{VC(G)}{2}$. Thus we expect:
$VC(G)lg(n) + \binom{VC(G)}{2} + (n - VC(G))\binom{VC(G)}{2} \approx \binom{n}{2}$
$VC(G)lg(n) + (n + 1 - VC(G))\binom{VC(G)}{2} \approx \binom{n}{2}$

This leads to finding the roots of a cubic equation. The author is tempted to pull out a computer algebra system for this part, so we leave this as an exercise to the reader.

4 Further study

We have listed more than a few modern books on random graphs and the incompressibility method in the bibliography. Be warned that most books like [9],[3],[8],and [4] do not even discuss the incompressibility method. Instead, they rely on a technique known as the probabilistic method, by which they manipulate equations about expected distribution of graph properties. It would be a fruitfull endevor to replace these proofs with shorter and cleaner incompressibility ones.

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