

Lower Bound Visualization of a Zero-Sum Problem

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May 10, 2002

Abstract

Let $n(k, d)$ represent the number of points in $Z^d \bmod k$ needed to ensure that there exists a k -subset of them that adds to $0 \bmod k$. In this paper we present an algorithm that searches for maximal length sets having no k -subset adding to $0 \bmod k$ in Z^d , and gives the exact answer for $n(k, d)$ when the algorithm halts. Exact and lower bound sequences are analyzed for $n(k, d) | 1 \leq k \leq 9, 1 \leq d \leq 6$. AMS classification 11P Additive Number Theory

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1 Previous Work

Let $n(k, d)$ represent the number of points in $Z^d \bmod k$ needed to ensure that there exists a k -subset of them that adds to $0 \bmod k$.

INSERT APPLICATIONS HERE:

Over the years $n(k, d)$ has been referred to as the zero-sum problem, zero-sum set problem, Kitten problem[4], lattice point problem, Ein Extremalproblem fur Gitterpunkte[1], and who knows how many other names.

Results are usually extended to other additive finite abelian groups isomorphic to Z_k^d , and sometimes to applications such as Ramsey Theory[6].

Erdős, Ziv, and Ginzburg[2] are traditionally the fathers of this problem proving back in 1949 that $n(k, 1) = 2k - 1$.

Harborth[1] around 1973 came up with the bounds $k^d(k - 1) + 1 \geq n(k, d) \geq 2^d(k - 1) + 1$, and showed that $n(k, 2)$ achieves a lower bound when $k = 2^x 3^y$. In the early 1980's J.L Brenner[3] showed that $n(3,3)=19$ using a geometrical argument, Kemnitz[9] showed that $n(3,4)=41$, and Kemnitz made the conjecture that $n(k, 2) = 4k - 3$.

Recently, work has been done to find asymptotic constants, and prove the conjecture by Kemnitz that $n(k, 2) = 4k - 3$.

Alon and Dubiner[5] showed that $n(k, 2) \leq 6k - 5$.

Ronyai[10] proved if p is prime then $n(p, 2) \leq 4p - 2$.

Gao[7][8] has shown that:

If p is a prime then $n(p^i, 2) \leq 4p^i - 2$, and $n(k, 2) = 4k - 3$ for $k = 2^a 3^b 5^x 7^y m$ where a, b, x, y are non-negative integers and $m \leq (2^{a+2} 3^{b-1} 5^x 7^y)^{1/3}$.

We now search the space of small $n(k, d)$ for lower bound sequences.

2 Search Algorithm

To search the space of all k -subsets of size up to $n(k, d)$ we can use a brute force algorithm:

Let POINTS := all k^d modulo classes for $Z^d \text{ mod } k$

BruteForce[k,d]

For each set of size i

Choose all $|i|$ sets from POINTS (repetition allowed)

Check all k -subsets of d dimensions for zero-sums

$O(\text{BruteForce}[k,d]) = \sum_{k \leq i \leq n(k,d)} \binom{k^d + i - 1}{i} * \binom{i}{k} * d$

We can prune this search somewhat by noticing a few things: 1. Once a zero-sum sequence is found, you need not include it in larger sequences.

2. If a set has been tested for zero sum k -subsets, it need not be tested again as a subset of some larger set.

To implement this search we can view our search space as a k^d -ary tree. At each node we have k^d different points that we can choose from to add to our set. The goal is to find the longest paths from the root to depth $n(k, d) - 1$ in the tree that do not contain any zero-sum k -sets.

We search the tree in a depth first fashion recursively, while disregarding the choices at each node in further recursions.

When analyzing a set for zero-sum k -sets we avoid looking at the same subsets twice by not looking at k -sets already looked at the previous levels of the tree. Also, as an implementation note, memory for the recursions is pre-allocated to help prevent "thrashing" or rapid calls to the operating system to allocate and de-allocate memory.

This creates a non-polynomial, but more efficient algorithm:

Let POINTS := each of the k^d points in $Z^d \text{ mod } k$

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Let MAXSET := NULL

SumSearch[SET,POINTS]
  For each point left in POINTS
    Pick a point X from POINTS;
    Let SET = SET + X;
    If Size(SET) >= k
      If SET has no zero sum k-subsets
        SumSearch[SET, POINTS];
      Else
        If ( Size[MAXSET] < Size[SET-X] )
          MAXSET = SET-X;
    Remove X from POINTS;

```

3 Results

SumSearch[] was run on a dual Celeron 433mhz computer for values of $1 \leq k \leq 9$ and $1 \leq d \leq 6$ using up the better part of four months of CPU time.

This is a table of lower bounds computed for $n(k,d)$:

$$\leq n(k, d)$$

| | d=1 | d=2 | d=3 | d=4 | d=5 | d=6 |
|-----|-----|-----|-----|-----|-----|-----|
| k=1 | 1 | 1 | 1 | 1 | 1 | 1 |
| k=2 | 3 | 5 | 9 | 17 | 33 | 65 |
| k=3 | 5 | 9 | 19 | 41 | 77 | 141 |
| k=4 | 7 | 13 | 25 | 49 | 97 | 193 |
| k=5 | 9 | 17 | 37 | 74 | 150 | 257 |
| k=6 | 11 | 21 | 41 | 81 | 161 | 321 |
| k=7 | 13 | 25 | 55 | 107 | 193 | 385 |
| k=8 | 15 | 29 | 57 | 113 | 225 | 449 |
| k=9 | 17 | 33 | 65 | 129 | 257 | 513 |

A much more aesthetically pleasing table is the lower bounds computed for $n(k, d) - 1$, the maximal size of non-zero sets. We denote them as $g(k, d)$ where $g(k, d) = n(k, d) - 1$.

$$\leq g(k, d)$$

| | d=1 | d=2 | d=3 | d=4 | d=5 | d=6 |
|-----|-----|-----|-----|-----|-----|-----|
| k=1 | 0 | 0 | 0 | 0 | 0 | 0 |
| k=2 | 2 | 4 | 8 | 16 | 32 | 64 |
| k=3 | 4 | 8 | 18 | 40 | 76 | 140 |
| k=4 | 6 | 12 | 24 | 48 | 96 | 192 |
| k=5 | 8 | 16 | 36 | 73 | 149 | 256 |
| k=6 | 10 | 20 | 40 | 80 | 160 | 320 |
| k=7 | 12 | 24 | 54 | 106 | 192 | 384 |
| k=8 | 14 | 28 | 56 | 112 | 224 | 448 |
| k=9 | 16 | 32 | 64 | 128 | 256 | 512 |

Why are $g(5, 4)$ and $g(5, 5)$ prime and odd, when none of the others are?

The naive lower bound of Harborth $g(k, d) \geq 2^d(k - 1)$ can be achieved by constructing the sequence of all points with $\{0, 1\}$ coordinates and making $k - 1$ copies of each. Example:

$$g(3, 2) : \{00, 00, 10, 10, 01, 01, 11, 11\}$$

We will call this sequence Harborth's enumeration.

This is a table of the size of maximal non-zero sets minus the size of the Harborth enumeration.

$$\leq g(k, d) - 2^d(k - 1)$$

| | d=1 | d=2 | d=3 | d=4 | d=5 | d=6 |
|-----|-----|-----|-----|-----|-----|-----|
| k=1 | 0 | 0 | 0 | 0 | 0 | 0 |
| k=2 | 0 | 0 | 0 | 0 | 0 | 0 |
| k=3 | 0 | 0 | -2 | -8 | -12 | -12 |
| k=4 | 0 | 0 | 0 | 0 | 0 | 0 |
| k=5 | 0 | 0 | -4 | -9 | -21 | 0 |
| k=6 | 0 | 0 | 0 | 0 | 0 | 0 |
| k=7 | 0 | 0 | -6 | -10 | 0 | 0 |
| k=8 | 0 | 0 | 0 | 0 | 0 | 0 |
| k=9 | 0 | 0 | 0 | 0 | 0 | 0 |

The following sets exceeded the length of Harborth's enumeration:

$$g(3, 3), g(3, 4), g(3, 5), g(3, 6)$$

$$g(5, 3), g(5, 4), g(5, 5)$$

$$g(7, 3), g(7, 4)$$

Here an example of the maximal nonzero set $g(3, 3)$ where each point is repeated twice:

$$0, 0, 0$$

$$0, 0, 1$$

$$0, 1, 0$$

$$0, 1, 1$$

1, 0, 0
 1, 1, 0
 1, 2, 1
 1, 2, 2
 2, 2, 1

What about their structure allows them to have this extra "room" to grow beyond the Harboth enumeration length? Notice only $g(k, d)$ with odd values of k violated the Harboth enumeration length.

This table shows the size of maximal non-zero sets divided by $k-1$.

$$\frac{\leq g(k, d)}{k - 1}$$

| | d=1 | d=2 | d=3 | d=4 | d=5 | d=6 |
|-----|-----|-----|-----|---------|-------|-----|
| k=1 | 0 | 0 | 0 | 0 | 0 | 0 |
| k=2 | 2. | 4. | 8. | 16. | 32. | 64. |
| k=3 | 2. | 4. | 9. | 20. | 38. | 70. |
| k=4 | 2. | 4. | 8. | 16. | 32. | 64. |
| k=5 | 2. | 4. | 9. | 18.25 | 37.25 | 64. |
| k=6 | 2. | 4. | 8. | 16. | 32. | 64. |
| k=7 | 2. | 4. | 9. | 17.6667 | 32. | 64. |
| k=8 | 2. | 4. | 8. | 16. | 32. | 64. |
| k=9 | 2. | 4. | 8. | 16. | 32. | 64. |

Why are $g(5, 4)$, $g(5, 5)$, and $g(7, 4)$ the only ones not divisible by $k - 1$? All maximal length nonzero sets found are included in the appendix and at <http://www.public.iastate.edu/crb002/zeroSumSets.html>.

4 Future Work

Until constructive theorems about zero-sum sets are found, searching for maximal length non-zero sets is going to be a computationally "hard" problem. Randomized techniques might be helpful to find better lower bounds. Also, SumSearch[] is highly parallelizable and might make a good distributed computing task for proving small $n(k, d)$ and generating all their maximal length nonzero sum sets. The goal of this work was to build a collection of maximal length non-zero sets to keep current researchers informed, and introduce others to the problem. These results, along with all source code and data files will be made available at <http://www.public.iastate.edu/crb002/zeroSumSets.html>.

5 Acknowledgments

Thanks to Martin J. Erickson for writing a great introductory book on combinatorics[4] in which I discovered this problem, and Thomas L. Horine of Princeton who introduced me to Weidong Gao's work.

6 Appendix

The appendix is rather large, so to save paper maximal nonzero sets found were posted at <http://www.public.iastate.edu/crb002/zeroSumSets.html>.

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