Uniform Plane Wave Solution to Maxwell's Equations

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1 Helmholtz Equations

Solutions to the *Helmholtz equations* describe how electric $\mathbf{E}$ (V m^{-1}) and magnetic fields $\mathbf{H}$ (A m^{-1}) propagate (travel) through a homogeneous material. The Helmholtz equations for the time harmonic forms of the electric and magnetic fields $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ assuming an $e^{j\omega t}$ time dependence are

$$\nabla^2\tilde{\mathbf{E}} + k^2\tilde{\mathbf{E}} = 0 \qquad (1)$$

and

$$\nabla^2\tilde{\mathbf{H}} + k^2\tilde{\mathbf{H}} = 0 \qquad (2)$$

where: $k = \omega\sqrt{\mu\epsilon}$ is the wave number or propagation constant (rad m^{-1}); $\omega = 2\pi f$ is the radial frequency (rad s^{-1}); f is the frequency (s^{-1} or Hz); μ is the permeability of the material through which the wave is propagating (H m^{-1}); and ϵ is the permittivity of the material (F m^{-1}).

2 Uniform Plane Wave

An electric field of the form $\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\tilde{E}_x$ solves (1) if

$$\tilde{E}_x = E_{xo}\, e^{-jkz}. \qquad (3)$$

This is a *uniform plane wave* propagating in the $+\hat{\mathbf{z}}$ direction. Uniform plane waves have uniform (constant) properties in a plane perpendicular to their direction of propagation. For the uniform plane wave described by (3) the plane of uniformity is the x–y plane. E_{xo} is complex in general so it has a magnitude $|E_{xo}|$ and a phase $e^{j\phi}$. Since the electric and magnetic fields are coupled, there must be a magnetic field $\tilde{\mathbf{H}}(z)$ associated with this electric field. The magnetic field can be found using one of Maxwell's equations, specifically $\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$. The magnetic field has only a $\hat{\mathbf{y}}$ component and has the form:

$$\tilde{H}_y = \frac{E_{xo}}{\eta} e^{-jkz} \qquad (4)$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance of the material (Ω). Note that (4) also solves (2).

2.1 Lossless Medium

The real electric field is found by converting the time harmonic form of the electric field in (3) using the relationship $\mathbf{E}(t,z) = \Re\left\{\tilde{\mathbf{E}}\, e^{j\omega t}\right\}$ where $\Re\{\}$ takes the real part of the argument. The real electric field has the form

$$\mathbf{E}(t,z) = \hat{\mathbf{x}}\,|E_{xo}|\, cos\,(\omega t - kz + \phi)\,. \tag{5}$$

This is a wave equation with amplitude $|E_{xo}|$, radial frequency ω, phase constant or wave number $k = 2\pi/\lambda$, wavelength λ (m), and reference phase ϕ (rad) propagating in the $+\hat{\mathbf{z}}$ direction. Note that the magnitude of the complex field $\tilde{\mathbf{E}}(z)$ is the amplitude of the real field $\mathbf{E}(t,z)$ and the phase of $\tilde{\mathbf{E}}(z)$ is the spatial portion of the phase of $\mathbf{E}(t,z)$.

By taking the time derivative of the argument of the cosine function in (5) it can be shown that the phase velocity u_p is

$$u_p = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}. \tag{6}$$

The permeability μ and permittivity ϵ can also be written $\mu = \mu_r\mu_o$ and $\epsilon = \epsilon_r\epsilon_o$, respectively, where μ_r and ϵ_r are the relative permeability and permittivity of the material, and μ_o and ϵ_o are the permeability and permittivity of free space. Virtually all materials relevant to microwave remote sensing have relative permeabilities of unity. We will always assume that $\mu_r = 1$ and that $\mu = \mu_o$. The relative permittivities of materials relevant to microwave remote sensing can vary in magnitude from 1 to 80 and may also be complex. "Free space" is a term used for a vacuum. Air has a relative permittivity of approximately unity, so it can be approximated as free space. The phase velocity in free space, c, is found by substituting into (6).

$$c = \frac{\omega}{k_o} = \frac{1}{\sqrt{\mu_o\epsilon_o}} \tag{7}$$

Here k_o is the free space wave number. Evaluation of (7) gives $c = 2.9979 \times 10^8$ m s^{-1}.

Using (6), (7), and other definitions (e.g. $u_p = \lambda f$), the following relationships can be written:

$$k_o = \frac{\omega}{c} = \frac{2\pi}{\lambda_o} \tag{8}$$

$$\lambda = \frac{\lambda_o}{\sqrt{\epsilon_r}} \tag{9}$$

$$u_p = \frac{c}{\sqrt{\epsilon_r}} \tag{10}$$

$$k = \frac{\omega}{u_p} = \frac{\omega}{c/\sqrt{\epsilon_r}} = k_o\sqrt{\epsilon_r} \tag{11}$$

where $\lambda_o = c/f$ is the free space wavelength.

The general relationship between the electric and magnetic fields is given as follows:

$$\tilde{\mathbf{H}} = \frac{1}{\eta}\,\hat{\mathbf{k}} \times \tilde{\mathbf{E}} \tag{12}$$

$$\tilde{\mathbf{E}} = -\eta\,\hat{\mathbf{k}} \times \tilde{\mathbf{H}} \tag{13}$$

where $\hat{\mathbf{k}}$ denotes the direction of propagation.

2.2 Lossy Medium

When the medium through which an electromagnetic wave is propagating is *lossy*, the magnitude of the wave is attenuated. Lossy mediums have wave numbers k that are complex: $k = k' - jk''$, where $k' = \Re\{k\}$ is the real part of the wavenumber, and $k'' = \Im\{k\}$ is the imaginary part. To see the effect of a complex wavenumber, consider the real time–dependent field of the uniform plane wave in (3).

$$\mathbf{E}(t, z) = \Re\left\{\hat{\mathbf{x}}\, E_{xo}\, e^{-j(k'-jk'')z} e^{j\omega t}\right\} \tag{14}$$

The result is:

$$\mathbf{E}(t, z) = \hat{\mathbf{x}}\, |E_{xo}|\, e^{-k''\, z} \cos(\omega t - k'z + \phi) \tag{15}$$

The imaginary part of the wavenumber k'' attenuates the wave. The real part of the wavenumber k' acts just as before as the phase constant and determines the wavelength of the wave as it propagates in the medium, as well as the propagation velocity u_p.

Since we assume that $\mu = \mu_o$ ($\mu_r = 1$) in microwave remote sensing, then the relative permittivity ϵ_r of a lossy medium must also be complex: $\epsilon_r = \epsilon_r' - j\epsilon_r''$. Note that the relative permittivity is also sometimes called the *dielectric constant*. When $\mu_r = 1$,

$$k' = k_o |\Re\{\sqrt{\epsilon_r}\}| \tag{16}$$

$$k'' = k_o |\Im\{\sqrt{\epsilon_r}\}|. \tag{17}$$

When evaluating $\sqrt{\epsilon_r}$ care must be taken since ϵ_r is a complex number. It is more convenient, and more physical, to define the *index of refraction* $n = \sqrt{\epsilon_r} = n' - jn''$. Using this definition,

$$k' = k_o |\Re\{n\}| = k_o n' \tag{18}$$

$$k'' = k_o |\Im\{n\}| = k_o n'' \tag{19}$$

The refractive index and relative permittivity are related as follows:

$$\epsilon_r' = (n')^2 - (n'')^2 \tag{20}$$

$$\epsilon_r'' = 2n'n'' \tag{21}$$

$$n' = \sqrt{\frac{\sqrt{(\epsilon_r')^2 + (\epsilon_r'')^2} + \epsilon_r'}{2}} \tag{22}$$

$$n'' = \sqrt{\frac{\sqrt{(\epsilon_r')^2 + (\epsilon_r'')^2} - \epsilon_r'}{2}} \tag{23}$$

The advantage of using the index of refraction to characterize a medium instead of the relative permittivity (dielectric constant) is shown by (20) through (23). When using the relative permittivity, the phase constant k' is a function of *both* the real *and imaginary* parts of ϵ_r. Furthermore, the attenuation k'' is *also* a function of both the real and imaginary parts of ϵ_r. If we use the refractive index, then the phase constant k' is only a function of n', and the attenuation k'' is only a function of n''. Hence the real part of the refractive index determines the wavelength of the wave in the medium ($\lambda = \lambda_o/n'$) and the phase velocity ($u_p = c/n'$), and the imaginary part of the refractive index determines the attenuation ($k'' = k_o n''$).

The magnitude of the electric field in a lossy medium can be found from (15).

$$|\tilde{\mathbf{E}}| = |E_{xo}|\, e^{-k''\, z} \tag{24}$$

When $|\tilde{\mathbf{E}}|$ is $e^{-1} = 37\%$ of its original magnitude $|E_{xo}|$, then $z = \delta_s = 1/k''$, where δ_s is called the *skin depth* of the material. The skin depth is a measure of how far a wave can propagate into a medium. Waves are attenuated quickly in a medium that has a small skin depth. When the medium is lossless ($k'' = 0$) then the skin depth is infinite and the wave propagates indefinitely. Metals have high electrical conductivities, and hence ϵ_r'' is large, which generally leads to a large k''. Metals therefor have very small skin depths.

3 Two–component Fields

Coming soon.

4 Polarization

Coming soon.

5 Propagation of Power

The Poynting vector $\mathbf{S}$ is defined as

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{25}$$

and has units of W m^{-2}. It represents the instantaneous power per unit area carried by an electromagnetic wave. The direction of the Poynting vector is $\hat{k}$, the direction of propagation. The time–average power density is

$$\mathbf{S_{av}} = \frac{1}{2}\, \Re\left\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\right\} \tag{26}$$

where * denotes the complex conjugate. *The time–average power density is the quantity that is measured by a microwave radiometer.* In passive remote sensing, this time–average power is used to infer specific properties of the source of the electromagnetic radiation. In reality, only a specific component of the time–average power is measured as determined by the polarization of the radiometer's antenna.

For a wave propagating in $\hat{z}$ in a lossless medium ($\Im\{n\} = 0$),

$$\mathbf{S_{av}}(z) = \hat{z}\, \frac{|\tilde{\mathbf{E}}|^2}{2\eta} = \hat{z}\, \frac{|E_{xo}|^2 + |E_{yo}|^2}{2\eta} \tag{27}$$

For a wave propagating in $\hat{z}$ in a lossy medium ($\Im\{n\} \neq 0$),

$$\mathbf{S_{av}}(z) = \hat{z}\, \frac{|\tilde{\mathbf{E}}(z=0)|^2}{2|\eta|} \cos(\theta_\eta)\, e^{-2\,k''\, z} \tag{28}$$

where $\eta = |\eta| e^{j\theta_\eta}$.

6 Nepers and Decibels

As electromagnetic radiation travels through a lossy medium, it is attenuated according to the electrical properties of the medium, specifically the imaginary part of the refractive index, n. For a uniform plane wave propagating in $+\hat{z}$,

$$S_{av}(z) = \frac{1}{2}\,\Re\left\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\right\} = \frac{|\tilde{\mathbf{E}}(z=0)|^2}{2|\eta|}\cos(\theta_\eta)\,e^{-2\,k''\,z} \tag{29}$$

where again $k'' = k_o n''$. At any point $z = z'$, the ratio of the time–average power at $z = z'$, $S_{av}(z')$, to the time–average power at $z = 0$, $S_{av}(0)$, is:

$$\frac{S_{av}(z')}{S_{av}(0)} = \frac{\frac{|\tilde{\mathbf{E}}(z=0)|^2}{2|\eta|}\cos(\theta_\eta)\,e^{-2\,k''\,z'}}{\frac{|\tilde{\mathbf{E}}(z=0)|^2}{2|\eta|}\cos(\theta_\eta)\,e^{0}} = e^{-\alpha z'} \tag{30}$$

where $\alpha = 2\,k''$ is the attenuation constant of the medium.

In general, any power ratio like the one in (30) can be expressed in units of either nepers (Np) or decibels (dB). Consider a power ratio P_2/P_1 where $P_2 < P_1$ (the power has been attenuated). To find the ratio in nepers, simply take the natural logarithm of the power ratio.

$$N = \ln\frac{P_2}{P_1} \tag{31}$$

N has units of Np and the two powers are related through

$$P_2 = P_1 e^{-|N|} \tag{32}$$

(Note that since $P_2 < P_1$, $N < 0$). To find the ratio in decibels, take the base–ten logarithm of the power ratio and multiply by ten.

$$N = 10\log\frac{P_2}{P_1} \tag{33}$$

Here N has units of dB. To convert between nepers and decibels, think of what a power ratio of $P_1/P_2 = e \approx 2.718$ would be in decibels:

$$N = \ln\frac{P_1}{P_2} = \ln(e) = 1\ \text{Np} = 10\log\frac{P_1}{P_2} = 10\log(e) \approx 4.343\ \text{dB} \tag{34}$$

Attenuation constants α will be given with units of either Np m^{-1} or dB m^{-1}. To compute the actual attenuation of power using (30), the units of dB m^{-1} must be converted to Np m^{-1} using (34).