

11/21/2012

Coordinate Systems and Vector Calculus

This is the common language that we need to share in order to communicate in class.

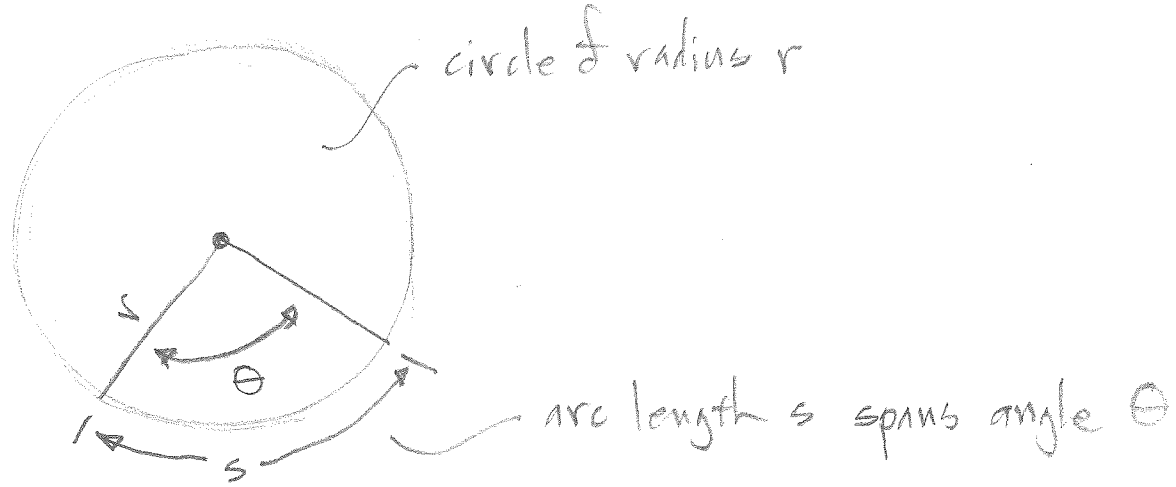
Electromagnetic fields vary in 3-D space.

There are 3 standard orthogonal coordinate systems that can be used: Cartesian; cylindrical; spherical.

The choice of which system to use is determined by the geometry of the problem: which system is most convenient?

See illustrations from the Uaby reference, especially figures 3-2, 3-8, 3-9, 3-10, 3-13, 3-14, Tables 3-1 and 3-2.

Definition of radian angle



$$\theta \equiv \frac{s}{r} = \frac{\text{arc length}}{\text{radius of arc}}, \quad [\theta] = \text{radians}$$

Example: $\theta_{\text{circle}} = \frac{2\pi r}{r} \overset{\text{circumference of a circle}}{=} = 2\pi \text{ rad} = 360^\circ$

Note that: $s = \text{arc length} = r\theta$ when $[\theta] = \text{radians}$!

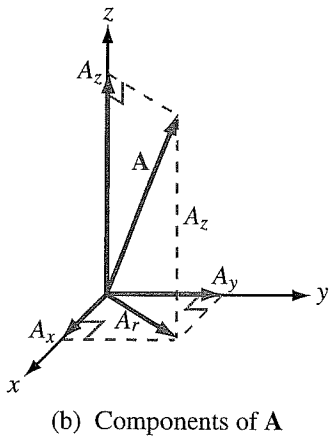
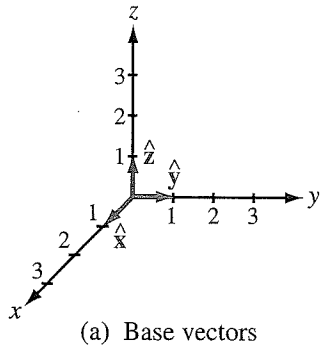
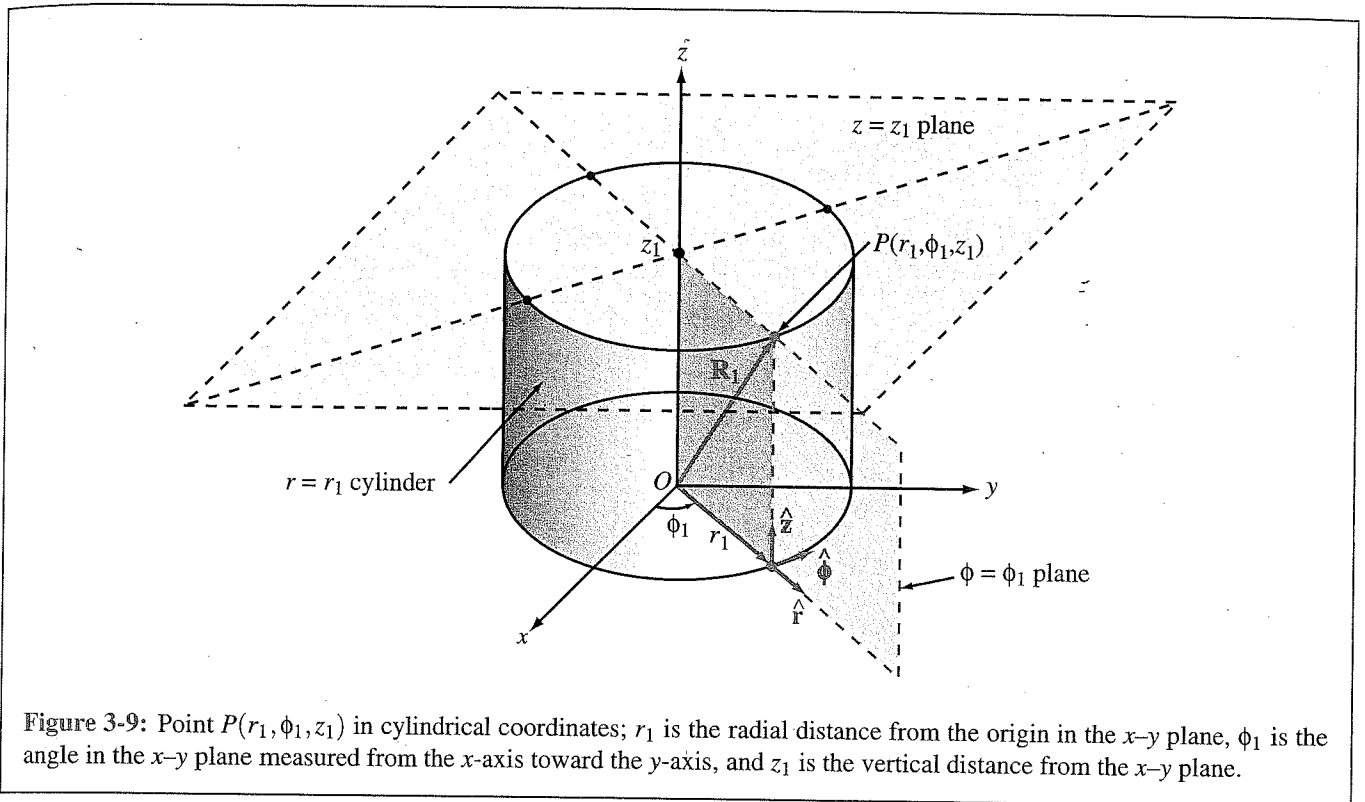
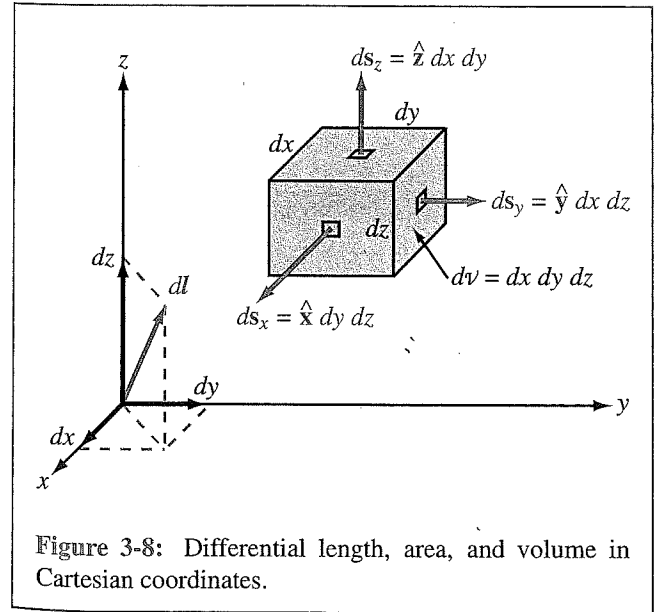


Figure 3-2: Cartesian coordinate system: (a) base vectors \hat{x} , \hat{y} , and \hat{z} , and (b) components of vector A.



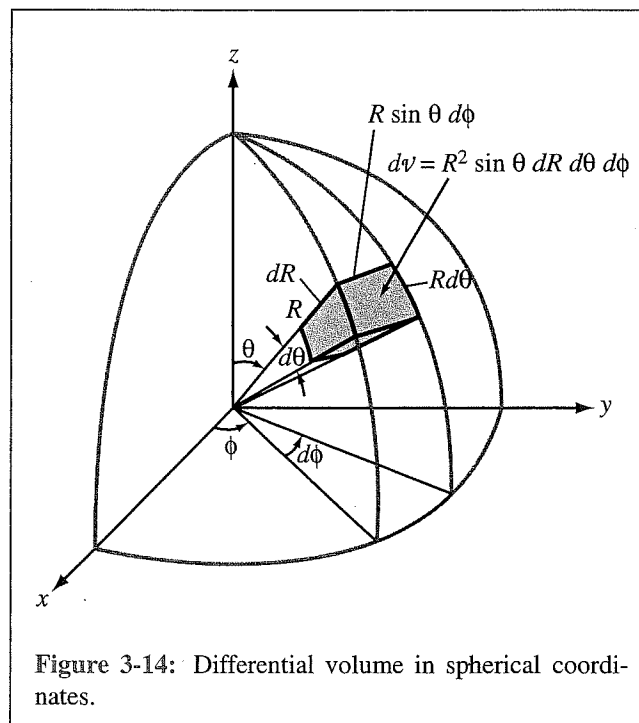
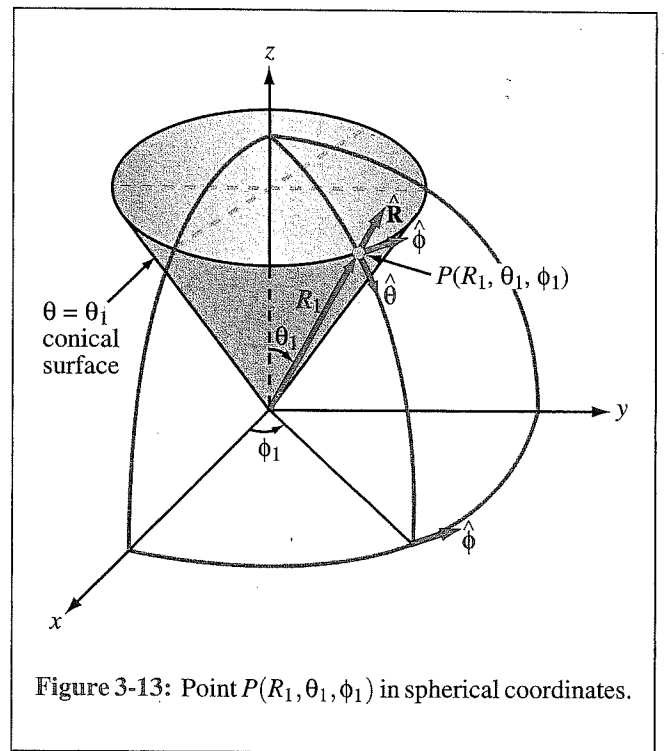
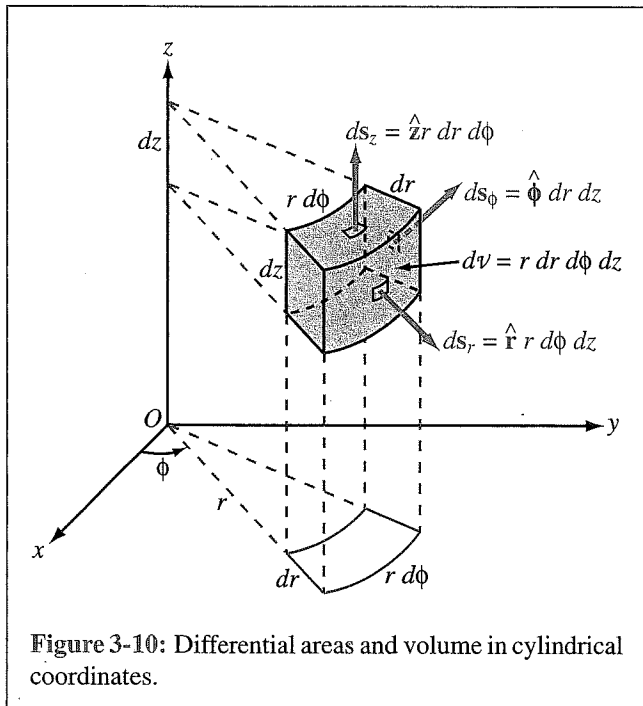


Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, $\mathbf{A} =$	$\hat{x}A_x + \hat{y}A_y + \hat{z}A_z$	$\hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$	$\hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$
Magnitude of \mathbf{A} , $ \mathbf{A} =$	$\sqrt{A_x^2 + A_y^2 + A_z^2}$	$\sqrt{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$ for $P(x_1, y_1, z_1)$	$\hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1$	$\hat{r}r_1 + \hat{z}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{R}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$ $\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$ $\hat{x} \times \hat{y} = \hat{z}$ $\hat{y} \times \hat{z} = \hat{x}$ $\hat{z} \times \hat{x} = \hat{y}$	$\hat{r} \cdot \hat{r} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1$ $\hat{r} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{z} = \hat{z} \cdot \hat{r} = 0$ $\hat{r} \times \hat{\phi} = \hat{z}$ $\hat{\phi} \times \hat{z} = \hat{r}$ $\hat{z} \times \hat{r} = \hat{\phi}$	$\hat{R} \cdot \hat{R} = \hat{\theta} \cdot \hat{\theta} = \hat{\phi} \cdot \hat{\phi} = 1$ $\hat{R} \cdot \hat{\theta} = \hat{\theta} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{R} = 0$ $\hat{R} \times \hat{\theta} = \hat{\phi}$ $\hat{\theta} \times \hat{\phi} = \hat{R}$ $\hat{\phi} \times \hat{R} = \hat{\theta}$
Dot product, $\mathbf{A} \cdot \mathbf{B} =$	$A_x B_x + A_y B_y + A_z B_z$	$A_r B_r + A_\phi B_\phi + A_z B_z$	$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, $\mathbf{A} \times \mathbf{B} =$	$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{R} & \hat{\theta} & \hat{\phi} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l} =$	$\hat{x} dx + \hat{y} dy + \hat{z} dz$	$\hat{r} dr + \hat{\phi} r d\phi + \hat{z} dz$	$\hat{R} dR + \hat{\theta} R d\theta + \hat{\phi} R \sin\theta d\phi$
Differential surface areas	$ds_x = \hat{x} dy dz$ $ds_y = \hat{y} dx dz$ $ds_z = \hat{z} dx dy$	$ds_r = \hat{r} r d\phi dz$ $ds_\phi = \hat{\phi} dr dz$ $ds_z = \hat{z} r dr d\phi$	$ds_R = \hat{R} R^2 \sin\theta d\theta d\phi$ $ds_\theta = \hat{\theta} R \sin\theta dR d\phi$ $ds_\phi = \hat{\phi} R dR d\theta$
Differential volume, $d\mathbf{v} =$	$dx dy dz$	$r dr d\phi dz$	$R^2 \sin\theta dR d\theta d\phi$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{r} = \hat{x} \cos\phi + \hat{y} \sin\phi$ $\hat{\phi} = -\hat{x} \sin\phi + \hat{y} \cos\phi$ $\hat{z} = \hat{z}$	$A_r = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r \cos\phi$ $y = r \sin\phi$ $z = z$	$\hat{x} = \hat{r} \cos\phi - \hat{\phi} \sin\phi$ $\hat{y} = \hat{r} \sin\phi + \hat{\phi} \cos\phi$ $\hat{z} = \hat{z}$	$A_x = A_r \cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{R} = \hat{x} \sin\theta \cos\phi + \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta$ $\hat{\theta} = \hat{x} \cos\theta \cos\phi + \hat{y} \cos\theta \sin\phi - \hat{z} \sin\theta$ $\hat{\phi} = -\hat{x} \sin\phi + \hat{y} \cos\phi$	$A_R = A_x \sin\theta \cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta \cos\phi + A_y \cos\theta \sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$
Spherical to Cartesian	$x = R \sin\theta \cos\phi$ $y = R \sin\theta \sin\phi$ $z = R \cos\theta$	$\hat{x} = \hat{R} \sin\theta \cos\phi + \hat{\theta} \cos\theta \cos\phi - \hat{\phi} \sin\phi$ $\hat{y} = \hat{R} \sin\theta \sin\phi + \hat{\theta} \cos\theta \sin\phi + \hat{\phi} \cos\phi$ $\hat{z} = \hat{R} \cos\theta - \hat{\theta} \sin\theta$	$A_x = A_R \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi$ $A_y = A_R \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$
Cylindrical to spherical	$R = \sqrt{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{R} = \hat{r} \sin\theta + \hat{z} \cos\theta$ $\hat{\theta} = \hat{r} \cos\theta - \hat{z} \sin\theta$ $\hat{\phi} = \hat{\phi}$	$A_R = A_r \sin\theta + A_z \cos\theta$ $A_\theta = A_r \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R \sin\theta$ $\phi = \phi$ $z = R \cos\theta$	$\hat{r} = \hat{R} \sin\theta + \hat{\theta} \cos\theta$ $\hat{\phi} = \hat{\phi}$ $\hat{z} = \hat{R} \cos\theta - \hat{\theta} \sin\theta$	$A_r = A_R \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \hat{\mathbf{r}} \frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}} \frac{1}{r} \frac{\partial V}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (rA_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{\mathbf{z}} \frac{1}{r} \left[\frac{\partial}{\partial r} (rA_\phi) - \frac{\partial A_r}{\partial \phi} \right]$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \hat{\mathbf{R}} \frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}} \frac{1}{R} \frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{R \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R \sin \theta) A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}} \frac{1}{R \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] + \hat{\boldsymbol{\theta}} \frac{1}{R} \left[\frac{1}{\sin \theta} \frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R} (RA_\phi) \right] + \hat{\boldsymbol{\phi}} \frac{1}{R} \left[\frac{\partial}{\partial R} (RA_\theta) - \frac{\partial A_R}{\partial \theta} \right]$$

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$

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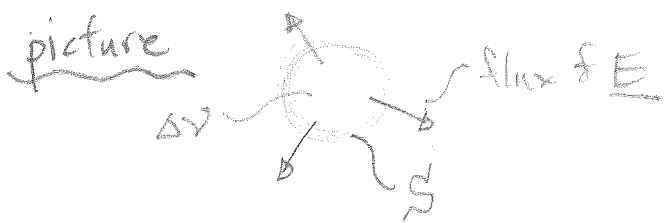
Divergence and Curl

Fig 3-20
in Vlaby ref.

The nature of electromagnetic radiation requires special mathematical notation, like divergence and curl.

the divergence of a vector field \underline{E} at a point

$$= \text{div } \underline{E} \cong \lim_{\Delta V \rightarrow 0} \left\{ \frac{\oint_S \underline{E} \cdot d\underline{s}}{\Delta V} \right\} = \text{the net outward flux of the vector field } \underline{E} \text{ per unit volume } \Delta V \text{ across a closed surface } S \text{ that encloses } \Delta V.$$



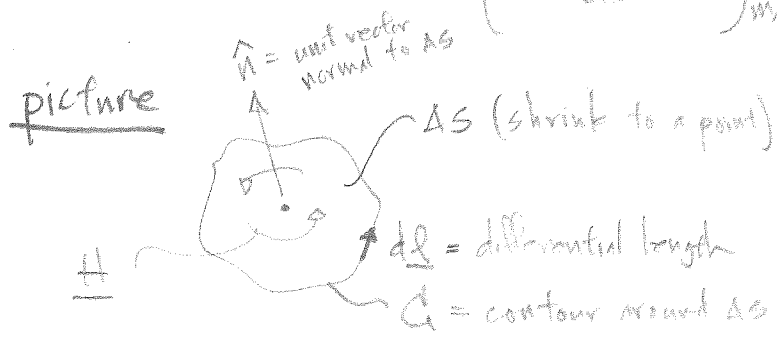
see
handout for
other coordinate
systems

notation: $\nabla \cdot \underline{E} \cong \text{div } \underline{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$ in Cartesian coordinates

where $\nabla =$ the gradient operator $\cong \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$ in Cartesian coordinates
and $\underline{E} = \hat{x} E_x + \hat{y} E_y + \hat{z} E_z$

the curl of a vector field \underline{H} at a point

$$= \text{curl } \underline{H} \cong \lim_{\Delta S \rightarrow 0} \left\{ \frac{\hat{n} \oint_C \underline{H} \cdot d\underline{l}}{\Delta S} \right\}_{\text{max}}$$



the circulation of vector field \underline{H} per unit surface area ΔS . ΔS is oriented such that the circulation is maximum and \hat{n} is the normal to ΔS defined by right-hand rule.

Fig 3-23
in Vlaby

$$\nabla \times \underline{H} \cong \text{curl } \underline{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \text{ in Cartesian coordinates}$$

where $\underline{H} = H_x \hat{x} + H_y \hat{y} + H_z \hat{z}$

$$\nabla \times \underline{H} = \hat{x} \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) - \hat{y} \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) + \hat{z} \left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \right)$$

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The divergence theorem

Extending the definition of the divergence of a vector field \underline{E} from a differential volume to any volume:

$$\int_V \nabla \cdot \underline{E} \, d\tau = \oint_S \underline{E} \cdot d\underline{s} \quad \text{for any } \underline{E}$$

divergence of \underline{E} from enclosed volume V = flux of \underline{E} through surface S that encloses volume V

physical interpretation
Gauss' law: $\nabla \cdot \underline{D} = \rho_v$
 ρ_v = charge density, $C \cdot m^{-3}$
 $\int_V \nabla \cdot \underline{D} \, d\tau = \int_V \rho_v \, d\tau = Q_{\text{enclosed}}$
= total charge enclosed in V
= $\oint_S \underline{D} \cdot d\underline{s}$

We can determine the total amount of charge inside by observing the field lines that penetrate the surface!

(charge is only thing that could produce \underline{D})

Stokes' theorem

Extending the definition of the curl to any surface:

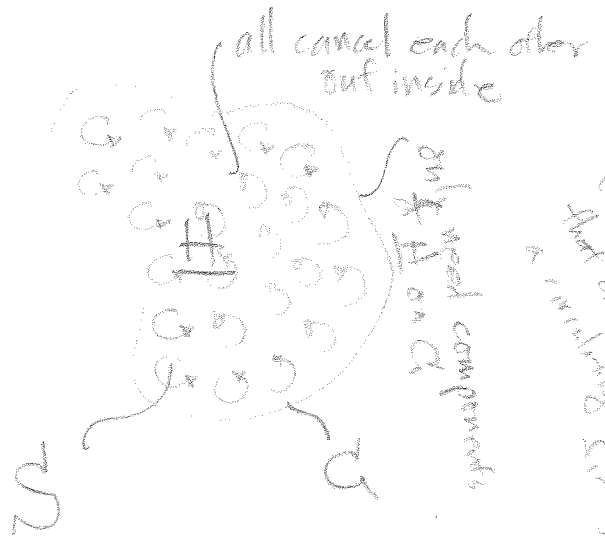
$$\int_S (\nabla \times \underline{H}) \cdot d\underline{s} = \oint_C \underline{H} \cdot d\underline{l} \quad \text{for any } \underline{H}$$

the curl of \underline{H} over any surface S = the line integral of \underline{H} along the contour C that encloses S

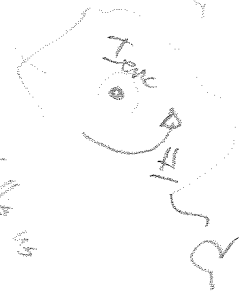
physical interpretation
Ampere's law: $\nabla \times \underline{H} = \underline{J}$

$$\int_S (\nabla \times \underline{H}) \cdot d\underline{s} = \int_S \underline{J} \cdot d\underline{s} = I_{\text{enclosed}}$$

$$= \oint_C \underline{H} \cdot d\underline{l}$$



(This is what law tells us that current produces a circulating magnetic field)



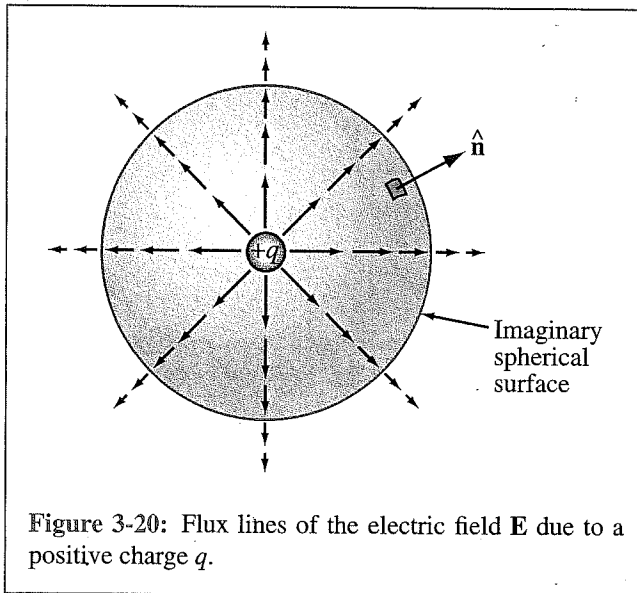


Figure 3-20: Flux lines of the electric field \mathbf{E} due to a positive charge q .

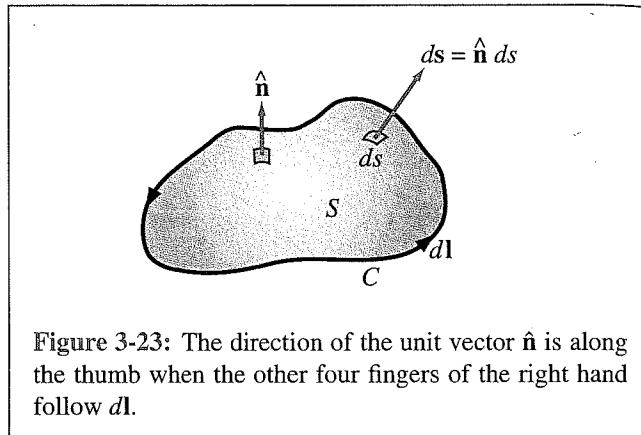


Figure 3-23: The direction of the unit vector $\hat{\mathbf{n}}$ is along the thumb when the other four fingers of the right hand follow $d\mathbf{l}$.

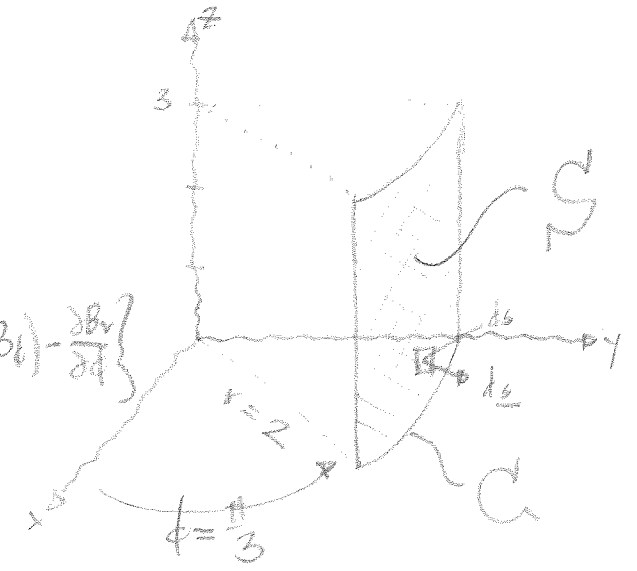
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Example Given $\underline{B} = \hat{z} \frac{\cos \phi}{r}$, verify Stokes' theorem for the cylindrical segment defined by $r=2, \frac{\pi}{3} \leq \phi \leq \frac{\pi}{2}, 0 \leq z \leq 3$.

Stokes' theorem

$$\int_S (\nabla \times \underline{B}) \cdot d\underline{s} = \oint_C \underline{B} \cdot d\underline{l}$$

$$\begin{aligned} \nabla \times \underline{B} &= \hat{r} \left\{ \frac{1}{r} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} \right\} + \hat{\phi} \left\{ \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right\} + \hat{z} \frac{1}{r} \left\{ \frac{\partial}{\partial r} (r B_\phi) - \frac{\partial B_r}{\partial \phi} \right\} \\ &= \hat{r} \left\{ \frac{1}{r} \left(-\frac{1}{r} \sin \phi \right) - 0 \right\} + \hat{\phi} \left\{ 0 - \left(-\frac{\cos \phi}{r^2} \right) \right\} + \hat{z} \frac{1}{r} \{ 0 - 0 \} \\ &= -\hat{r} \frac{\sin \phi}{r^2} + \hat{\phi} \frac{\cos \phi}{r^2} \end{aligned}$$



$$\nabla \times \underline{B} = \frac{1}{r^2} (\hat{\phi} \cos \phi - \hat{r} \sin \phi)$$

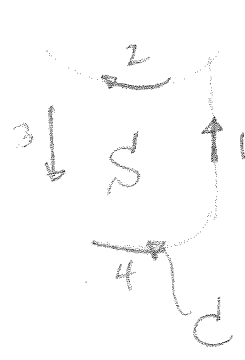
$$\begin{aligned} \int_S (\nabla \times \underline{B}) \cdot d\underline{s} &= \int_0^3 \int_{\pi/3}^{\pi/2} \left\{ \frac{1}{r^2} (\hat{\phi} \cos \phi - \hat{r} \sin \phi) \right\} \cdot \hat{r} r d\phi dz \Big|_{r=2} = \int_0^3 \int_{\pi/3}^{\pi/2} \frac{-\sin \phi}{r^2} r d\phi dz \Big|_{r=2} \\ &= -\frac{3}{2} \int_{\pi/3}^{\pi/2} \sin \phi d\phi = -\frac{3}{2} [-\cos \phi]_{\pi/3}^{\pi/2} = \frac{3}{2} (\cos \frac{\pi}{2} - \cos \frac{\pi}{3}) = -\frac{3}{4} \end{aligned}$$

$$\oint_C \underline{B} \cdot d\underline{l} \Rightarrow \textcircled{1} \underline{B} \Big|_{\phi=\frac{\pi}{2}, r=2} = \hat{z} \frac{\cos \frac{\pi}{2}}{2} = 0$$

$$\textcircled{2} \underline{B} \Big|_{z=3, r=2} = \hat{z} \frac{\cos \phi}{2}$$

$$\textcircled{4} \underline{B} \Big|_{z=0, r=2} = \hat{z} \frac{\cos \phi}{2}$$

$$\textcircled{3} \underline{B} \Big|_{\phi=\frac{\pi}{3}, r=2} = \hat{z} \frac{\cos \frac{\pi}{3}}{2} = \hat{z} \frac{1}{4}$$



but $d\underline{l} = r d\phi \hat{\phi}$ for $\textcircled{2}$ and $\textcircled{4}$
and $\hat{\phi} \cdot \hat{z} = 0$ so both $\textcircled{2}$ and $\textcircled{4}$ contribute nothing to $\oint_C \underline{B} \cdot d\underline{l}$!

$$\oint_C \underline{B} \cdot d\underline{l} = \textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4} = \int_0^3 \left(\hat{z} \frac{1}{4} \right) \cdot \left(\hat{z} dz \right) = \int_0^3 \frac{dz}{4} = \frac{z}{4} \Big|_0^3 = \frac{3}{4}$$

yes!