Maxwell's equations in free space (time harmonic form)

\[ \nabla \cdot \mathbf{E} = 0 \]
\[ \nabla \times \mathbf{E} = -j \omega \mu \mathbf{H} \]
\[ \nabla \cdot \mathbf{H} = 0 \]
\[ \nabla \times \mathbf{H} = j \omega \varepsilon \mathbf{E} \]

contains effect of electrical conductivity

Helmholtz equations (or homogeneous wave equations)

\[ \nabla^2 \mathbf{E} + \omega^2 \varepsilon \mathbf{E} = 0 \]
\[ \nabla^2 \mathbf{H} + \omega^2 \mu \mathbf{H} = 0 \]

Electromagnetic radiation must satisfy these conditions!

Solution to Helmholtz eqn

We know electromagnetic radiation propagates as a wave. What does a wave look like in time harmonic form?

A \cos(\omega t - \beta z) - wave traveling in +z direction (\omega t)

\[ = \text{Re} \{ A [ \cos(\omega t - \beta z) + j \sin(\omega t - \beta z)] \} \]
\[ = \text{Re} \{ A e^{-j(\omega t - \beta z)} \} = \text{Re} \{ A e^{-j\beta z} e^{j\omega t} \} \]
\[ = \text{Re} \{ A e^{-j\beta z} \} \text{reminds us that } e^{j\omega t} \text{ dependence is assumed} \]

complex field representation

Does this wave solve (1) and (2)?
Given $\mathbf{E} = \mathbf{E}_0 e^{-a k z}$, show that this complex field solves (1), then find the corresponding $\mathbf{H}$ field, and finally the real fields $\mathbf{E}(t)$ and $\mathbf{H}(t)$.

(1) $\nabla^2 \mathbf{E} + \omega^2 \mu \varepsilon \mathbf{E} = 0$

$$\nabla^2 \mathbf{E} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E}(x, y, z) = \mathbf{E}_0 \left( \frac{\partial^2}{\partial x^2} e^{-a k z} + \frac{\partial^2}{\partial y^2} e^{-a k z} + \frac{\partial^2}{\partial z^2} e^{-a k z} \right)$$

$$= \mathbf{E}_0 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) e^{-a k z}$$

$$= \mathbf{E}_0 \left( -a^2 k^2 e^{-a k z} \right)$$

$$= -a^2 k^2 \mathbf{E}_0 e^{-a k z}$$

Substitute into (1):

$$\nabla^2 \mathbf{E} + \omega^2 \mu \varepsilon \mathbf{E} = 0$$

$$= -a^2 k^2 \mathbf{E}_0 e^{-a k z} + \omega^2 \mu \varepsilon \mathbf{E}_0 e^{-a k z} = 0$$

Just a constant

6 Propagation in + \omega direction (r.h.s.)

7 Component in \omega direction

Find $\mathbf{H}$ in 3D

$$\nabla \times \mathbf{E} = -\mu \varepsilon_0 \frac{\partial \mathbf{H}}{\partial t} = \mathbf{E}_0 \left( \frac{\partial}{\partial x} \sin \frac{\omega t}{c} - \frac{\omega}{c} \frac{\partial}{\partial z} \cos \frac{\omega t}{c} \right) + \mathbf{E}_0 \left( \frac{\partial}{\partial y} \sin \frac{\omega t}{c} - \frac{\omega}{c} \frac{\partial}{\partial z} \cos \frac{\omega t}{c} \right)$$

$\mathbf{E}$ has no $y$ or $z$ components,

$$= \hat{\mathbf{y}} \frac{\partial \mathbf{E}_0}{\partial z} = \hat{\mathbf{y}} \frac{\partial}{\partial z} \left( \mathbf{E}_0 e^{-a k z} \right) = \frac{\mathbf{E}_0}{\varepsilon} \left( \frac{\partial}{\partial z} + a k \right) e^{-a k z} = \frac{\mathbf{H}_0}{\varepsilon} e^{-a k z}$$

$$\mathbf{H} = \frac{\mathbf{H}_0}{\eta} e^{a k z}$$

$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \text{intrinsic impedance of material}$
Real fields

\[ E = \mathbf{x} \cdot \mathbf{E}_0 \cdot e^{-j k x} \]

\[ E(t) = \text{Re} \left\{ \mathbf{E}_0 e^{j \omega t} \right\} = \text{Re} \left\{ \mathbf{x} \cdot \mathbf{E}_0 e^{j \omega t} \cdot e^{j k x} \right\} = \mathbf{x} \cdot \mathbf{E}_0 \cdot \text{Re} \left\{ e^{j \omega t} \cdot e^{j k x} \right\} = \mathbf{x} \cdot \mathbf{E}_0 \cdot \text{Re} \left\{ e^{j (\omega t - k x)} \right\} \]

\[ E(t) = \mathbf{x} \cdot \mathbf{E}_0 \cdot \cos(\omega t - k x) \quad \text{our old wave equation!} \]

\[ \lambda = \frac{2\pi}{k} = \text{angular frequency}, \quad \text{rad} \]
\[ k = \omega \sqrt{\mu \varepsilon} = \text{wavenumber,} \quad \text{rad/m} \]
\[ \frac{c}{\lambda} = \frac{c}{\omega} = \frac{c}{\sqrt{\mu \varepsilon}} = \text{phase velocity,} \quad \text{m/s} \]
\[ \lambda = \frac{2\pi}{k} = \frac{C}{\sqrt{\varepsilon}} = \text{wavelength,} \quad \text{m} \]

In free space, \( \varepsilon = \varepsilon_0 \) and \( \mu = \mu_0 \), so

\[ \frac{C}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\left(4\pi \cdot 10^{-7} \cdot 1 \right) \left(8.85 \cdot 10^{-12} \cdot 1 \right)}} = 2.998 \cdot 10^8 \text{m/s} \]

\[ c \equiv \text{speed of light in a vacuum!} \]

In a material, \( \varepsilon = \varepsilon \varepsilon_0 \) and \( \mu = \mu \mu_0 / \mu_0 = \mu_0 \)

\[ \frac{C}{\sqrt{\varepsilon_0 \mu_0}} \quad \text{ask what happens when} \; \varepsilon \; \text{increases} \]
\[ \lambda_0 = \frac{C}{\varepsilon_0 \mu_0} \quad \text{free-space wavelength} \]
\[ \lambda = \frac{\lambda_0}{\sqrt{\varepsilon_0 \mu_0}} \quad \text{ask what happens when} \; \varepsilon \; \text{increases} \]
\[ k_0 \equiv \frac{2\pi}{\lambda_0} = \text{free-space wavenumber} = \frac{C}{\lambda_0} \]
\[ k = k_0 \frac{C}{\sqrt{\varepsilon_0 \mu_0}} \]
\[ H = \hat{\gamma} \frac{E_0^+}{i} e^{-\delta k z} \]

\[ H(t) = \text{Re} \left\{ H e^{jat} \right\} = \text{Re} \left\{ \frac{E_0^+}{i} e^{-\delta k z} e^{jat} \right\} \]

\[ H(t) = \frac{E_0^+}{i} \cos (at - k z), \quad E(t) = \frac{E_0^+}{k} \sin (at - k z) \]

Radiation

Recall from Faraday's law and the Maxwell-Ampère law that the coupling of \( E \) and \( H \) required accelerating charges: \( \frac{\Delta \mathbf{E}}{\Delta t} \neq \text{constant} \).

*Accelerating charges produce electromagnetic radiation, or radiate.*

Radiation propagates as a spherical wave (which can be described in spherical coordinates).

At points far from the accelerating charge, the wave front of a spherical wave can be approximated by a plane wave.

A uniform plane wave is characterized by electric and magnetic fields that have uniform properties at all points across an infinite plane.

\[ E = \hat{x} E_0^+ e^{-j k z}, \quad H = \frac{E_0^+}{k} e^{-j k z} \]

both propagate in the \( x \)-plane of uniform properties in the \( xy \)-plane.

General relationship between \( E \) and \( H \) of a propagating wave.

For a uniform plane wave propagating in an arbitrary direction \( \mathbf{k} \),

\[ H = \frac{1}{k} \mathbf{k} \times E, \quad E = -j (k \mathbf{k} \times H) \]

where \( k \) could be \( \phi \) or \( \theta \) or \( \mathbf{k} \), etc.; \( \mathbf{k} = \text{propagation vector}, |k| = k, \quad k = \frac{\omega}{c} \]

*Note: Fields can not propagate in the direction of propagation.*
Figure 7-3: Waves radiated by an EM source, such as a light bulb or an antenna, have spherical wavefronts, as in (a); to a distant observer, however, the wavefront across the observer's aperture appears approximately planar, as in (b).