Optimal buffer size and dynamic rate control for a queueing network with impatient customers in heavy traffic. *

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Abstract

We address a rate control problem associated with a single server Markovian queueing system with customer abandonment in heavy traffic. The controller can choose a buffer size for the queuing network and also can dynamically control the service rate (equivalently the arrival rate) depending on the current state of the network. An infinite horizon cost minimization problem is considered here. The cost function includes a penalty for each rejected customer, a control cost related to the adjustment of the service rate and a penalty for each abandoned customer. We obtain an explicit optimal strategy for the limiting diffusion control problem (the Brownian control problem or BCP) which consists of a threshold-type optimal rejection process and a feedback-type optimal drift control. This solution is then used to construct an asymptotically optimal control policy, i.e. an optimal buffer size and an optimal service rate for the queueing network in heavy traffic. The properties of generalized regulator maps and weak convergence techniques are employed to prove the asymptotic optimality of this policy. In addition, we identify the parameter regimes where the infinite buffer size is optimal.

Abbreviated Title: Controlled queues with reneging.

1 Introduction.

In this article, we address a stochastic control problem associated with a single server queueing system in heavy traffic. The controller can choose and fix a buffer size of the queue (= maximum number of customers allowed to wait in the queue) and can dynamically control the arrival and/or the service rates depending on the current state of the network. Our results are applicable in the two extreme cases where only the service rate is allowed to control or else only the arrival rate is controlled, as well as in the more general situation where both the arrival and service rates are controlled subjected to the heavy traffic constraints described in (2.2)-(2.5) below. It is assumed that

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the inter-arrival times and the service times are independent and identically distributed. Another feature of this queueing model is that each admitted customer has a memoryless "patience clock" which "rings" after a random time. This random time is assumed to be exponentially distributed and independent of all the other system variables. The customer abandons the system if the service is not completed before the clock rings. Such models are referred to as queueing networks with reneging or systems with impatient customers (see [25, 31, 30, 26, 27, 32]). Here we consider a system optimization problem related to a queueing system with reneging in heavy traffic parameter regime. In this model, higher queue-length leads to higher abandonment costs regardless of the actions of the controller. Therefore, the controller would like to achieve a shorter queue-length by exercising the available controls: choosing a higher service rate (compared to the arrival rate,) incurring a higher control cost or choosing a smaller buffer-size (i.e reject more customers) leading to a higher rejection penalty. These considerations lead to three different cost components in our infinite horizon discounted cost criterion: A penalty for each rejected customer, a control cost related the adjustment of the arrival and service rates as well as a penalty for each abandoned customer. Under the heavy traffic assumptions on the system parameters, we show how to approximate this controlled queueing system by a Brownian network with reneging. This Brownian system is described by a solution to a controlled stochastic differential equation, whose drift and diffusion coefficients are obtained as the limiting values of the parameters of the queueing system. We first obtain an explicit optimal strategy for the Brownian control problem (BCP). It consists of an optimal drift rate and an optimal rejection policy. This optimal rejection policy turned out to be of a threshold-type and hence, it corresponds to an optimal buffer size. Second, we use this optimal strategy for the BCP to construct a policy for choosing the buffer size and arrival and/or service rates for the queueing network control problem in heavy traffic and prove its asymptotic optimality. Depending on the system parameters as well as the cost structure, we observe that the infinite buffer size can also be optimal in certain cases (for example, it is optimal when the rejection penalty per customer is prohibitively high), and we identify the corresponding parameter regimes for such situations.

The study of Markovian queueing systems with customer abandonment was initiated by Palm [25] who modeled impatient customer behavior in a telephone switchboard. In a series of recent articles [31, 30, 26, 27], Ward and coauthors addressed different aspects of such queueing networks with impatient customers. We refer to [27] for a detailed list of references on this topic. In [27], it is generalized to more general non-Markovian models with customer abandonment in heavy traffic and they identify the limiting Brownian network and characterize the stationary distribution. All the above articles address the issue of *performance evaluation* of such queueing systems. There are numerous articles that address the issue of *system optimization* of different types of queues in heavy traffic (see [5, 8, 9, 6, 14, 15, 18, 3, 29] etc.) without customer abandonment. Recently, [32] addressed a *system optimization* problem in the context of a GI/GI/1 queue with impatient customers in heavy traffic. The authors use the solution of the approximating BCP to construct an asymptotically optimal admission control policy. There, the arrival and service rates of the queueing network are constants and heavy traffic assumptions imply that they are equal. In fact, their choice of an admission control policy is equivalent to the choice of a rejection policy in this article. Their threshold-type optimal admission control policy indeed provides a finite optimal buffer-size for the queueing network. *Performance evaluation* of queueing network models in heavy traffic with variable arrival and service rates have been studied in [22, 23, 34] etc. For a fixed buffer-size, a similar rate control problem for a BCP which minimizes a long term average cost is studied in [2]. More recently in [13], the authors considered a BCP with variable rates and a general customer rejection policy (variable buffer size). An optimal Markovian strategy which consists of a feed-back type drift parameter and an optimal finite buffer size were derived there. The article [13] also provides a numerical method to determine optimal buffer size for such systems. The two articles [32] and [13] provides the main motivation for this work. Here we consider a Markovian queueing system in heavy traffic with impatient customers, where the controller can choose and fix the buffer size of the network as well as dynamically control the arrival and service rates to minimize a certain cost function.

In our controlled queueing network where the arrival and service rates are allowed to depend on the state of the system, we first need to define a notion of heavy traffic (similar to that in [34]) and obtain the BCP as a formal limit of the queueing system under diffusion scaling. These state-dependent controlled arrival and service rates are the so called *marginally* state-dependent rates(see [21], also referred as "thin controls" in [1], [4]) and they satisfy a heavy traffic condition specified in Section 2.2 (see (2.5) and Remark 2.3 (a)). In the BCP, the controlled drift of the stochastic differential equation is analogous to the difference between the arrival and the service rates and the controlled increasing process is analogous to the rejection process which represents the cumulative number of rejected customers from the queue. In the BCP, we allow the rejection process to be any adapted, right continuous non-decreasing process, which includes all the threshold-type rejection policies that correspond to the finite buffer situation. Our results on the BCP describe an explicit optimal feedback type drift control process. Furthermore, we identify the necessary and sufficient condition for the existence of a finite optimal buffer length. In our cost criterion, $p > 0$ denote the revenue lost per rejection, $\gamma > 0$ denote the customer reneging rate, $\beta > 0$ represents the cost for each reneging customer (such as a refund given to these dissatisfied customers (as in [32])) and $\delta > 0$ is a discount factor (can be thought of as the bank interest rate). Let $p_0 = \frac{\beta\gamma}{(\delta+\gamma)}$. We show that when $0 < p < p_0$, there is an optimal rejection policy associated with a finite buffer size b_p^*, whereas for $p \geq p_0$, optimal rejection process is identically zero (i.e. not rejecting any customer is optimal). It also turned out that the threshold p_0 is independent of the control cost $C(\cdot)$ (see (2.4)). Next, we show that our controlled queueing network under heavy traffic can be approximated by the above described Brownian network. Hence, using the solution to the BCP, we propose a candidate for an asymptotically optimal strategy which consists of a buffer size as well as dynamically controlled arrival and/or service rates. Using the properties of generalized regulator maps (see [32] and references therein) and the weak convergence methods, we will be able to prove the asymptotic optimality of this strategy when $0 < p < p_0$ and when $p > p_0$. We also conclude that the finite optimal buffer size exists when $0 < p < p_0$ and that infinite buffer size is optimal if $p > p_0$. When $0 < p < p_0$ and if there is no drift control (fixed arrival and service rates which are assumed to be equal to each other) then the optimal finite buffer size (equivalently, an optimal admission control policy) is obtained in [32]. Their optimal strategy for the BCP and the asymptotically optimal policy for the queueing network problem are in agreement with the results we obtained here.

The paper is organized as follows: Section 2 has the problem description including the details of the queueing network, the cost structure for the control problem as well as the main result of the article. In Section 3 we discuss the approximating BCP and obtain its explicit solution. The BCP addressed here is a singular stochastic control problem and it can be read independent of the other sections. Section 4 begins with a short discussion of generalized regulator maps, which will be used later in the proofs of the theorems that follow. Rest of this section is devoted to proving the main theorem. Throughout this article, all the processes are assumed to be in the space $\mathcal{D}([0,\infty), I\!R^k)$ ($\equiv$ The space of right continuous functions with left limits) for some $k \geq 1$ and we use " $\Rightarrow$ " to represent the weak convergence of the processes in the usual Skorokhod topology.

2 Problem Description and the Main Result.

2.1 Model Formulation.

We consider a sequence of queueing networks in heavy traffic indexed by $n \geq 1$. Each network is equipped with adjustable arrival and/or service rates and possibly a finite buffer size. The job of a "controller" is to choose these state-dependent rates as well as the buffer size so that an infinite horizon discounted cost structure (see (2.14)) is minimized. In addition, we have customer abandonment in this model, which cannot be controlled. Thus, the control structure here is represented by $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \equiv (\underset{\sim}{\lambda} = \{\lambda_n(\cdot)\}, \underset{\sim}{\mu} = \{\mu_n(\cdot)\}, b_n)$, where λ_n, μ_n are functions of the current queue-length representing the state dependent arrival and service rates and $b_n \equiv \sqrt{n}b$ ($b = \infty$ is allowed) for some $b > 0$, is the buffer size for the n-th network, satisfying some admissibility conditions (see Definition 2.2).

We assume that all the processes defined for the queueing network are defined on some common probability space. For $n \geq 1$, the dynamics of the nth network under a control $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ is described below. We assume that initially the queue is empty. The arrival time for the first customer is exponentially distributed with rate $\lambda_n(0)$ and the server immediately starts serving this customer. At this instant, the queue-length is 1 and the required service time to complete service to the first customer and the time until the second customer arrives is assumed to be independent and exponentially distributed with rates $\lambda_n(1)$ and $\mu_n(1)$ respectively. In addition, this customer can abandon the queue if the service is not completed within a random amount of time (patience time), which is assumed to be exponentially distributed with rate γ_n. We call a time instant an "event-time" if at that instant, either a new customer arrives or an existing customer leaves because of service completion or abandonment. At any "event-time", if the current queue-length is k, where $k \geq 0$, we assume the following memoryless structure: (remaining) inter-arrival time for the next customer, (remaining) service time for the current customer being served and (remaining) patience time for *each* of the existing customers in the queue are independent and distributed as exponential random variables with rates $\lambda_n(k)$, $\mu_n(k)I_{\{k>0\}}$ and γ_n respectively. In addition, if the buffer size $b_n = \sqrt{n}b$ is finite, then every incoming customer is rejected when the current queue-length is $b_n = \sqrt{n}b$, and no customer is rejected if $b = \infty$ is chosen. One can also think of the value b as an admission control threshold where the customers are allowed to join the queue only if the queue length is less than b(See [32]). We assume that the server does not idle unless the buffer is empty (queue-length is zero). The sequence in which available jobs in the queue are served is irrelevant because of our Markovian structure. Figure 1 describes the dynamics of the n-th queueing network ($n \geq 1$) at any time point $t \geq 0$.

A more rigorous description of our model is as follows: Let $Q_n(t)$ denote the queue-length process at time t, $t \geq 0, n \geq 1$. We assume $Q_n(0) = 0$ and $\{Q_n(t) : t \geq 0\}$ is a jump-Markov process with state space $\mathbb{Z}^+$ (= *set of all non-negative integers*) and jump intensities are given by

$$q^n_{k,k+1} = \lambda_n(k)I_{\{k<\sqrt{n}b\}},\ q^n_{k,k-1} = \mu_n(k)I_{\{k>0\}} + k\gamma_n,\ k \in \mathbb{Z}^+,$$

and $q^n_{k',\ell'} = 0$ for all other values of $k', \ell' \in \mathbb{Z}^+$. It is well-known (see Chapter 6 of [20]) that such a process can be represented as a linear combination of time-changed independent Poisson processes:

$$Q_n(t) = Y^A_n\left(\int_0^t \bar{\lambda}_n(Q_n(s))ds\right) - Y^S_n\left(\int_0^t \bar{\mu}_n(Q_n(s))ds\right) - Y^R_n\left(\int_0^t \gamma_n Q_n(s)ds\right),\ t \geq 1, \quad (2.1)$$

Figure 1: Dynamics of the n-th queueing network

where $\bar{\lambda}_n(k) \doteq \lambda_n(k) I_{\{k<\sqrt{n}b\}}$, $\bar{\mu}_n(k) \doteq \mu_n(k) I_{\{k>0\}}$ are the "effective" rates, and Y_n^A, Y_n^S, Y_n^R are independent Poisson processes with intensities 1. We will use (2.1) as the definition of the queue-length process in our model (see [34] for similar queueing models with state dependent rates).

2.2 Heavy traffic and admissible controls.

First we state our assumption on the reneging rates. See [32] for a similar assumption (see also Remark 2.3 (a) below).

Assumption 2.1 *There exists $\gamma > 0$ such that*

$$n\gamma_n \to \gamma > 0 \text{ as } n \to \infty$$

We assume that the system operates under heavy-traffic (i.e. the long-run-average arrival and service rates are equal), under any *admissible* control policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ that the controller chooses.

Definition 2.2 (Admissible Controls) *A control $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \equiv (\{\lambda_n(\cdot)\}, \{\mu_n(\cdot)\}, b)$ is called admissible for the queueing network if $\lambda_n(\cdot), \mu_n(\cdot)$ are nonnegative, continuous functions defined on $[0, \infty)$ and $b \in (0, \infty]$ such that for some λ, and $\mu > 0$ we have the following:*

(i) For some constant $c > 0$,

$$\sup_{n\geq 1} \sup_{x\geq 0} [\lambda_n(x) \vee \mu_n(x)] \leq c, \quad \inf_{n\geq 1} \inf_{x\geq 0} [\lambda_n(x) \wedge \mu_n(x)] > 0, \tag{2.2}$$

(ii)

$$\sup_{x\geq 0} |\lambda_n(x) - \lambda| \to 0, \quad \sup_{x\geq 0} |\mu_n(x) - \mu| \to 0 \text{ as } n \to \infty. \tag{2.3}$$

(iii) For $n \geq 1$, define

$$u_n(x) \doteq \sqrt{n}(\mu_n(\sqrt{n}x) - \lambda_n(\sqrt{n}x)) \geq 0, \text{ for each } x \geq 0, \tag{2.4}$$

then $\{u_n(\cdot)\}$ *is a sequence of nonnegative, uniformly Lipschitz continuous (with a Lipschitz constant* κ_u*) and for some nonnegative function* $u(\cdot)$*,*

$$\sup_{x\geq 0}|u_n(x)-u(x)| \to 0 \text{ as } n\to\infty. \tag{2.5}$$

Remark 2.3 *(a) Assumption 2.1 guarantees that customer abandonment rates do not influence the long-run average departure rate. Parts (ii) and (iii) of the Definition 2.2 imply that the system is in "heavy traffic", i.e.*

$$\lambda = \mu. \tag{2.6}$$

(b) The lower bound on the rates assumed in (2.2) and their continuity guarantee that the representation of queue-length in (2.1) is possible (see [20]).

(c) As it is often the case in heavy-traffic analysis of queueing networks, (because of an underlying functional central limit theorem) the diffusion scaled queue-length

$$\hat{Q}_n(\cdot) = \frac{Q_n(n\cdot)}{\sqrt{n}},\ n\geq 1,\ t\geq 0. \tag{2.7}$$

stabilizes. This is the reason for studying the asymptotic behavior of the system and the associated cost criterion (see (2.14) below.) under the diffusion scaling.

Since we study the system under diffusion scaling, we consider buffer sizes of the order $\sqrt{n}$ *for the* n*-th queueing network. It is possible to consider a more general situation, with the buffer sizes* b_n *for the* n*-th queue and it can be shown that any "reasonable" policy will have to satisfy* $b_n/\sqrt{n}\to b\in(0,\infty]$ *along some subsequence (see Lemma 9 of [32]). To reduce the notational overhead, we simply take* $b_n=\sqrt{n}b, b\in(0,\infty]$ *in this paper.*

(d) Note that in our setup, any admissible policy will affect the system behavior (in diffusion scale) marginally, via $u_n(\cdot)$*. We call this the "marginal drift function" and its limiting version* $u(\cdot)$ *as the "asymptotic marginal drift function" for a given* $(\underset{\sim}{\lambda},\underset{\sim}{\mu},b)$*.*

From the properties of the marginal drift functions in (2.4) -(2.5) in Definition 2.2, we conclude that $u(\cdot)$ *is also a nonnegative Lipschitz continuous function with the same Lipschitz constant* κ_u*.*

(e) For a concrete example of rates satisfying all our admissibility conditions in Definition 2.2, consider the following class of rates:

$$\lambda_n(x)=\lambda+\frac{1}{\sqrt{n}}u_1(\frac{x}{\sqrt{n}})+\frac{1}{n}v_1^n(\frac{x}{\sqrt{n}}),\ \mu_n(x)=\lambda+\frac{1}{\sqrt{n}}u_2(\frac{x}{\sqrt{n}})+\frac{1}{n}v_2^n(\frac{x}{\sqrt{n}}),\ x\geq 0,\ n\geq 1,$$

where $\lambda>0$, $u_1(\cdot)\leq u_2(\cdot)$ *are any two Lipschitz continuous functions with Lipschitz constants* $\kappa_1>0$*, and* $\kappa_2>0$ *respectively. Furthermore,* $\sup_{x\geq 0}v_i^n(x)=o(\sqrt{n})$ *for* $i=1,2$*.*

2.3 Scaled Processes.

First we define the lower- and upper-"reflection" processes: For $n\geq 1$

$$L_n(t)\doteq\mu_n(0)\int_0^t I_{\{Q_n(s)=0\}}ds,\qquad U_n(t)\doteq\lambda_n(\sqrt{n}b)\int_0^t I_{\{Q_n(s)=\sqrt{n}b\}}ds,\ \ t\geq 0. \tag{2.8}$$

This combined with (2.1), yields that

$$\begin{aligned}Q_n(t) &= \left[Y_n^A\left(\int_0^t \lambda_n(Q_n(s))ds\right) - \int_0^t \lambda_n(Q_n(s))ds\right] \\ &\quad - \left[Y_n^S\left(\int_0^t \mu_n(Q_n(s))ds\right) - \int_0^t \mu_n(Q_n(s))ds\right] - \left[Y_n^R\left(\int_0^t \gamma_n Q_n(s)ds\right) - \int_0^t \gamma_n Q_n(s)ds\right] \\ &\quad - \int_0^t [\mu_n(Q_n(s)) - \lambda_n(Q_n(s)) + \gamma_n Q_n(s)]ds + L_n(t) - U_n(t), \qquad (2.9)\end{aligned}$$

for all $n \geq 1$ and $t \geq 0$. Next, we define the following diffusion scaled Poisson processes:

$$\hat{Y}_n^A(t) = \frac{1}{\sqrt{n}}(Y_n^A(nt) - nt), \hat{Y}_n^S(t) = \frac{1}{\sqrt{n}}(Y_n^S(nt) - nt), \hat{Y}_n^R(t) = \frac{1}{\sqrt{n}}(Y_n^R(nt) - nt),\ t \geq 0. \quad (2.10)$$

and the diffusion-scaled versions of the reflection processes in (2.8) are given by:

$$\begin{aligned}\hat{L}_n(t) &\doteq \frac{1}{\sqrt{n}}L_n(nt) = \sqrt{n}\mu_n(0)\int_0^t I_{\{\hat{Q}_n(s)=0\}}ds,\ t \geq 0, \\ \hat{U}_n(t) &\doteq \frac{1}{\sqrt{n}}U_n(nt) = \sqrt{n}\lambda_n(\sqrt{n}b)\int_0^t I_{\{\hat{Q}_n(s)=b\}}ds,\ t \geq 0. \qquad (2.11)\end{aligned}$$

Using (2.7) and the definitions in $\hat{u}_n(\cdot)$, (2.4), (2.9), (2.10) and (2.11), one can easily verify that the following identity holds: For each $t \geq 0$,

$$\hat{Q}_n(t) = \hat{W}_n(t) - \int_0^t [u_n(\hat{Q}_n(s)) + n\gamma_n\hat{Q}_n(s)]ds + \hat{L}_n(t) - \hat{U}_n(t), \quad (2.12)$$

where,

$$\hat{W}_n(t) \doteq \hat{Y}_n^A\left(\int_0^t \bar{\lambda}_n(\sqrt{n}\hat{Q}_n(s))ds\right) - \hat{Y}_n^S\left(\int_0^t \bar{\mu}_n(\sqrt{n}\hat{Q}_n(s))ds\right) - \hat{Y}_n^R\left(\int_0^t \gamma_n\sqrt{n}\hat{Q}_n(s)ds\right). \quad (2.13)$$

2.4 The cost structure and the main result.

Note that in diffusion scaling, when the diffusion-scaled queue-length is $\hat{Q}_n(t)$ at any time $t \geq 0$, the customers abandon the queue at the collective rate of $n\gamma_n\hat{Q}_n(t)$. As mentioned in the Remark 2.3, our objective here is to study the asymptotic performance of the network under diffusion scaling. We assume that the cost of each abandoning customer is a constant $\beta > 0$, cost of controlling marginal rates is given by "a control cost function" $C(\cdot)$, and the income lost due to each rejected customer is a constant amount $p > 0$. We impose the following assumption on the control cost function $C(\cdot)$:

Assumption 2.4 (Control cost) *$C(\cdot)$ is a twice continuously differentiable function with $C(0) = C'(0) = 0$ and $C''(x) > 0$ for all $x \geq 0$.*

We consider an infinite horizon, discounted cost criterion, i.e. for any admissible policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$, we define the associated asymptotic cost by

$$J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \doteq \liminf_{n\to\infty} E\int_0^\infty e^{-\delta t}\left\{\left[\beta(n\gamma_n)\hat{Q}_n(t) + C(u_n(\hat{Q}_n(t)))\right]dt + p\, d\hat{U}_n(t)\right\}, \quad (2.14)$$

where $\delta > 0$ is a constant discount factor. The control problem here is to find an asymptotically optimal policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ which minimizes the cost defined in (2.14) among all the admissible policies. In other words, the problem is to find $J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*)$ such that

$$J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*) = \inf J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b),$$

where the infimum on the right side is taken over all the admissible policies $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ (as in Definition 2.2).

Definition 2.5 (A candidate for optimality) *Let $p > 0$, $\mathcal{V}_p$ and b_p^* be as in (3.38) and (3.39) of Section 3. Define $u_p^*(\cdot) = (C')^{-1}(\mathcal{V}_p^*(\cdot))$. Choose any two functions $\theta_1^*(\cdot)$, and $\theta_2^*(\cdot)$ defined on $[0, \infty)$, such that*

$$0 \leq \theta_2^*(x) - \theta_1^*(x) = u_p^*(x), \quad \textit{for all } x \geq 0.$$

Define

$$\lambda_n^*(x) \doteq \lambda + \frac{1}{\sqrt{n}}\theta_1^*\left(\frac{1}{\sqrt{n}}x\right), \quad \textit{and} \quad \mu_n^*(x) = \mu + \frac{1}{\sqrt{n}}\theta_2^*\left(\frac{1}{\sqrt{n}}x\right). \tag{2.15}$$

Then, $(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b_p^*) \equiv (\{\lambda_n^*(\cdot)\}_{n\geq 1}, \{\mu_n^*(\cdot)\}_{n\geq 1}, b_p^*)$ is a candidate for an optimal policy. The admissibility and asymptotic optimality of this policy will be shown in the proof of Theorem 2.6.

Theorem 2.6 (main result) *Our proposed policy $(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b_p^*)$ in Definition 2.5 is asymptotically optimal, i.e.*

$$J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b_p^*) \leq J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$$

for any admissible policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$.

Remark 2.7 *(a) We suppress the parameter $p > 0$ in $\underset{\sim}{\lambda}^*$ and $\underset{\sim}{\mu}^*$ for simplicity of the notation.*

(b) Also notice that, in the above proposed optimal policy, our arrival and service rates in (2.15) are not unique, even if the optimal asymptotic drift function $u^*(\cdot)$ is unique. This general setup covers more realistic special cases. For example, if the $\lambda > 0$ is a given constant and if $\lambda_n(x) \equiv \lambda$ for all x and the control problem is to choose an optimal state-dependent service rate $\mu_n(\cdot)$, then $\theta_1^* \equiv 0$ and $\theta_2^* \equiv u^*$, $\mu_n^*(\cdot)$ will be an optimal solution. Similarly, if $\mu > 0$ is given and $\mu_n(x) \equiv \mu$, then choosing $\theta_1^* = -u^*$, one can obtain an optimal state-dependent arrival rate λ_n^* for this problem.

3 Brownian control problem.

In this section, we describe a diffusion model that approximates the behavior of the queueing model under the diffusion scaling. The associated diffusion control problem is usually referred to as the Brownian control problem (BCP). From the functional central limit theorem for the Poisson processes (with unit intensity), it follows that

$$(\hat{Y}_n^A, \hat{Y}_n^S, \hat{Y}_n^R) \Rightarrow (W^A, W^S, W^R) \text{ as } n \to \infty,$$

where W^A, W^S, W^R are independent standard Brownian motions defined on some filtered probability space (see (4.24)). Intuitively, this suggests that from (2.13) and the definition of an admissible control $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ (Definition 2.2) that

$$\hat{W}_n \Rightarrow W$$

where W is a Brownian motion with zero drift and infinitesimal variance 2λ. In Lemma (4.5), we will verify this assertion. Also, from the definition of $\hat{L}_n, \hat{U}_n$ in (2.11), it is clear that these processes start from the origin, they are nondecreasing and increases only when $\hat{Q}_n = 0$ or b respectively. Thus, if $u(\cdot)$ is the associated asymptotic marginal drift function of $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$, one expects that the limit of diffusion scaled queues for each admissible policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ will satisfy:

$$X(t) = W(t) - \int_0^t [u(X(s)) + \gamma X(s)]ds + L(t) - U(t), \;\; t \geq 0,$$

where (X, L, U) is a weak limit of $(\hat{Q}_n, \hat{L}_n, \hat{U}_n)$. As it is the case in many queueing network control problems, studying the diffusion control problem with a cost structure similar to that in the queueing control problem often provides insights for the search of an asymptotically optimal control policy for the queueing control problem. In this section, we introduce the associated Brownian control problem and establish a result (Theorem 3.7) which explicitly derives an optimal control strategy.

In this section, the positive constants δ, β, γ, p and the function $C(\cdot)$ are as in Section 2.

We consider a state process $X_x(\cdot)$ which is a weak solution to the

$$X_x(t) = x - \int_0^t u(s)ds - \gamma \int_0^t X_x(s)ds + W(t) + L(t) - U(t), \;\; t \geq 0. \tag{3.1}$$

Where $x \geq 0$, $\{W(t) : t \geq 0\}$ is a one-dimensional Brownian motion, with no drift and variance 2λ, adapted to a right-continuous Brownian filtration $\{\mathcal{F}_t : t \geq 0\}$ on some probability space $(\Omega, \mathcal{F}, P)$. The σ-algebra $\mathcal{F}_0$ is assumed to contain all the null sets in $\mathcal{F}$. The processes $u(\cdot)$ and $U(\cdot)$ are the control processes and they satisfy the following:

The drift control process $\{u(t) : t \geq 0\}$ is progressively measurable with respect to $\{\mathcal{F}_t\}$, nonnegative and takes values in a control set $\mathcal{A} \subseteq [0, \infty)$ which will satisfy the Assumption 3.2 below. To make sure that the equation (3.1) makes sense, we will also assume

$$E \int_0^T u(s)ds < +\infty, \text{ for all } T > 0. \tag{3.2}$$

The singular control process $U(\cdot)$ is adapted to $\{\mathcal{F}_t\}$, nondecreasing, right-continuous with left limits and $U(0) = 0$. These processes also satisfy the property that the associated state process $X_x(\cdot)$ in (3.1) always remain nonnegative.

The other non-decreasing process $L(\cdot)$ represents the local-time process of $X_x(\cdot)$ at the origin. Therefore

$$\int_0^T I_{\{X_x(s) > 0\}} dL(s) = 0, \text{ for all } T > 0. \tag{3.3}$$

Definition 3.1 (Brownian control problem (BCP)) *For any given $x \geq 0$, any nonnegative solution $X_x(\cdot)$ to (3.1) together with the associated controls $u(\cdot)$ and $U(\cdot)$, which satisfy the above assumptions yield an admissible control system. More precisely, $((\Omega, \mathcal{F}, P), \mathcal{F}_t\}, X_x(\cdot), u(\cdot), U(\cdot))$ is called an admissible control system. With a slight abuse of notation, we simply write (X_x, u, U) for an admissible control policy. For such an admissible control policy (X_x, u, U), we define the cost criterion*

$$\widetilde{J}_p(x, u, U) \doteq E \int_0^\infty e^{-\delta t}[(\beta\gamma X_x(t) + C(u(t)))dt + p\, dU(t)]. \tag{3.4}$$

The value function of the control problem is defined by

$$\mathrm{V}_p(x) = \inf \widetilde{J}_p(x, u, U), \tag{3.5}$$

where the infimum above is taken over all admissible control policies $(X_x(\cdot), u(\cdot), U(\cdot))$.

Note that the value function also depends on the other parameters of the system such as δ, β, γ etc, but we suppress this dependence in our notation for the clarity of the presentation. In the next assumption, we define a critical value for the cost parameter p as well as specify some properties of the control set $\mathcal{A}$.

Assumption 3.2 (Control set) *Let,*

$$p_0 = \frac{\beta\gamma}{(\delta + \gamma)}. \tag{3.6}$$

We assume that the control set $\mathcal{A}$ is a priori known to the controller and it contains the interval $[0, \theta_0)$ where θ_0 is the unique positive real number which satisfies

$$C'(\theta_0) = p_0. \tag{3.7}$$

Remark 3.3 *(a) Notice that from Assumption 2.4, $C'(\cdot)$ is a strictly increasing function and hence from (3.7), for each $0 < p < p_0$ there exists a unique θ_p such that $C'(\theta_p) = p$.*

(b) In Section 4, we also use the notation W_x to denote $x + W$, where W is as in (3.1) and $x \geq 0$. In that case, W_x will denote a Brownian motion similar to W in all respect, except that it starts from $x \geq 0$.

Now we state the formal connections of the processes in the Definition 3.1 above to the processes introduced in the Section 2: The process $X_x(t)$ represents the diffusion limit of the queue-length process at time t, such that at time $t = 0$, the (diffusion-scaled) queue-length is equal to $x \geq 0$. The controller can choose the state-dependent drift rate function $u(\cdot)$ from the control set $\mathcal{A}$. The drift rate is analogous to the scaled difference between the service and the arrival rates in the queueing network (see (2.4) and (2.5) in Definition 2.2). We do not restrict to feedback-type drift-control in the BCP, and $u(\cdot)$ is any progressively measurable process which satisfies (3.2). However, the optimal drift turns out to be of the feed-back type. The other control $U(t)$ is analogous to the cumulative number of customers rejected from the queueing system during the time-interval $[0, t]$, for all $t \geq 0$. A trivial choice of such U is the identically zero function which is associated with the infinite buffer length situation. In such a situation, the controller makes no effort to reduce the queue-length process by rejecting customers and this can be a good control policy if the penalty for rejecting the customers is prohibitively high. Later in this section, we will show the optimality of the no rejection policy under such circumstances. A more interesting choice for U corresponds to a finite buffer situation, which rejects customers if the queue-length exceeds a pre-determined threshold $b > 0$ (the buffer-length). This case corresponds to $U(\cdot)$ being the local-time process of $X_x(\cdot)$ at the buffer-length $b > 0$. In general, this "rejection process" U can be chosen from any criteria (with jumps allowed) to reduce the queue-length (and need not be a local-time process), as far as it satisfies the constraints in the Definition 3.1 above.

Before we discuss the solution of the BCP in next two subsections, we introduce the following two functions Φ and Ψ which are essential in finding an optimal control policy. Introduce the function Φ on $[0, \infty)$ by

$$\Phi(y) \doteq \sup_{a \in \mathcal{A}} [ay - C(a)] \text{ for } \quad y \geq 0, \tag{3.8}$$

where $\mathcal{A}$ is the control set. Clearly $\Phi(y)$ is finite for each $y \geq 0$. For each $y \in [0, p_0]$, the supremum in (3.8) achieved at a unique point $\Psi(y) \in \mathcal{A}$, where

$$\Psi(y) = (C')^{-1}(y), \text{ for } 0 \leq y \leq p_0. \tag{3.9}$$

Note that, with the assumption in (2.4), the function $\Psi(\cdot)$ is continuously differentiable. For a detailed discussion on the properties of Φ and Ψ and their use in a discrete-time optimal control problem, we refer to [12]. In [2] and in [13], these functions were used in the construction of the optimal drift control processes and we follow the same approach here. In all these articles, these functions are denoted by ϕ and ψ (instead of Φ and Ψ respectively), but to distinguish these from the conventional Skorohod maps (which will be described in the Subsection 4.1), we intend use this different notation in this article. By Assumption 2.4, Ψ is strictly increasing on $[0, p_0]$. Furthermore, for each $0 < p \leq p_0$,

$$0 \leq \Psi(y) \leq \theta_p, \text{ when } 0 \leq y \leq p, \text{ where } C'(\theta_p) = p. \tag{3.10}$$

By (3.8) and (3.9), we obtain,

$$\Phi(y) = y\Psi(y) - C(\Psi(y)), \text{ for each } 0 \leq y \leq p_0, \tag{3.11}$$

and

$$\Phi'(y) = \Psi(y) \text{ for each } 0 \leq y \leq p_0. \tag{3.12}$$

3.1 A Verification Lemma

With the help of Φ in (3.8), the formal Hamilton-Jacobi-Bellman (HJB) equation (see [11]) for the BCP can be written as

$$\min \left\{ \frac{1}{2}\mathcal{V}''(x) - \Phi(\mathcal{V}'(x)) - \gamma x \mathcal{V}'(x) - \delta \mathcal{V}(x) + \beta\gamma x, \ \mathcal{V}'(x), \ p - \mathcal{V}'(x) \right\} = 0, \tag{3.13}$$

for almost every $x \in [0, \infty)$. The following verification lemma enables us to sort out an optimal strategy.

Lemma 3.4 (Verification Lemma) *Let $p > 0$ and $\mathcal{V}$ be a $\mathcal{C}^2$-function which satisfies the HJB equation in (3.13) together with the boundary condition*

$$\mathcal{V}'(0) = 0. \tag{3.14}$$

Then

$$\mathrm{V}_p(x) \geq \mathcal{V}(x), \quad \textit{for all } x \geq 0,$$

where $\mathrm{V}_p(\cdot)$ *is the value function defined in (3.5).*

Remark 3.5 *Since* $\mathcal{V}$ *satisfies (3.13),* $\mathcal{V}$ *may depend on p, but we do not make that explicit in our notation, for the clarity of the presentation.*

Proof. Fix $T > 0$. We apply the generalized Itô's Lemma (see p.285 of [24], [13]) to $\mathcal{V}(X_x(T))e^{-\delta T}$ where X_x satisfies (3.1) and $T > 0$. We also need a localization procedure, hence we introduce the sequence of stopping times $\{\tau_N : N \geq 1\}$ by

$$\begin{aligned} \tau_N &= \inf\{t > 0 : X_x(t) \geq N\} \\ &= +\infty, \quad \text{if the above set is empty.} \end{aligned} \tag{3.15}$$

Since, $U(\cdot)$ is nondecreasing, by (3.1), it follows that $0 \leq X_x(t) \leq X_x(t-)$ for all $t \geq 0$. Hence, $0 \leq X_x(t) \leq N$ for all $0 \leq t \leq \tau_N$.

$$\begin{aligned} &\mathcal{V}(X_x(T \wedge \tau_N))e^{-\delta(T\wedge\tau_N)} \\ &\quad = \mathcal{V}(x) + \int_0^{T\wedge\tau_N} e^{-\delta s}\mathcal{V}'(X_x(s-))dW(s) \\ &\quad + \int_0^{T\wedge\tau_N} e^{-\delta s}\mathcal{V}'(X_x(s-))dL(s) - \int_0^{T\wedge\tau_N} e^{-\delta s}\mathcal{V}'(X_x(s-))dU(s) \\ &\quad + \int_0^{T\wedge\tau_N} e^{-\delta s}\left(\frac{1}{2}\mathcal{V}''(X_x(s-)) - u(s)\mathcal{V}'(X_x(s-)) - \gamma X_x(s-)\mathcal{V}'(X_x(s-)) - \delta\mathcal{V}(X_x(s-))\right)ds \\ &\quad + \sum_{0<s\leq T\wedge\tau_N} e^{-\delta s}\left[\Delta\mathcal{V}(X_x(s)) + \mathcal{V}'(X_x(s-))\Delta U(s)\right], \end{aligned} \tag{3.16}$$

where $\Delta\mathcal{V}(X_x(s)) \doteq \mathcal{V}(X_x(s)) - \mathcal{V}(X_x(s-))$ and $\Delta U(s) \doteq U(s) - U(s-)$. Since, $0 \leq \mathcal{V}'(x) \leq p$, notice that

$$|\Delta\mathcal{V}(X_x(s))| \leq p\,|X_x(s) - X_x(s-)| = p\,[U(s) - U(s-)].$$

Therefore, $\sum_{0<s\leq T\wedge\tau_N} e^{-\delta s}|\Delta\mathcal{V}(X_x(s))| \leq pU(T\wedge\tau_N) < +\infty$. Similarly,

$$0 \leq \sum_{0<s\leq T\wedge\tau_N} e^{-\delta s}|\mathcal{V}'(X_x(s-))|\Delta U(s) \leq pU(T\wedge\tau_N) < \infty.$$

Hence, we can write,

$$\begin{aligned} &-\int_0^{T\wedge\tau_N} e^{-\delta s}\mathcal{V}'(X_x(s-))dU(s) + \sum_{0<s\leq T\wedge\tau_N} e^{-\delta s}\left[\Delta\mathcal{V}(X_x(s)) + \mathcal{V}'(X_x(s-))\Delta U(s)\right] \\ &\qquad = -\int_0^{T\wedge\tau_N} e^{-\delta s}\mathcal{V}'(X_x(s-))dU^c(s) + \sum_{0\leq s\leq T\wedge\tau_N} e^{-\delta s}\Delta\mathcal{V}(X_x(s)) \\ &\qquad \geq -p\int_0^{T\wedge\tau_N} e^{-\delta s}dU^c(s) - p\sum_{0\leq s\leq T\wedge\tau_N} e^{-\delta s}\Delta U(s) = -p\int_0^{T\wedge\tau_N} e^{-\delta s}dU(s), \end{aligned} \tag{3.17}$$

where $U^c(\cdot)$ is the continuous part of the process $U(\cdot)$. Combining (3.16) with (3.17) and then using (3.3), (3.8) and (3.13) and taking expected value, we obtain

$$\begin{aligned} &E\left(e^{-\delta(T\wedge\tau_N)}\mathcal{V}(X_x(T\wedge\tau_N))\right) \\ &\qquad\qquad \geq \mathcal{V}(x) - E\int_0^{T\wedge\tau_N} e^{-\delta s}\left[\beta\gamma X_x(s-) + C(u(s))\right]ds - pE\int_0^{T\wedge\tau_N} e^{-\delta s}dU(s). \end{aligned} \tag{3.18}$$

By (3.13), we also obtain

$$E\left[e^{-\delta(T\wedge\tau_N)}|\mathcal{V}(X_x(T\wedge\tau_N))|\right] \leq E\left[\left(|\mathcal{V}(0)| + pX_x(T\wedge\tau_N)\right)e^{-\delta(T\wedge\tau_N)}\right]. \tag{3.19}$$

We intend to estimate $E\left[X_x(T\wedge\tau_N)e^{-\delta(T\wedge\tau_N)}\right]$. Notice that

$$0 \leq E\left[X_x(T\wedge\tau_N)e^{-\delta(T\wedge\tau_N)}\right] \leq \left[E(X_x(T\wedge\tau_N)^2)\right]^{\frac{1}{2}}\left[E(e^{-2\delta(T\wedge\tau_N)})\right]^{\frac{1}{2}}. \tag{3.20}$$

To estimate $E(X_x(T\wedge\tau_N)^2)$, we can apply the generalized Itô's lemma to $X_x(T\wedge\tau_N)^2$ and follow a similar computation as in the derivation of (3.17) and eliminate the negative terms to obtain

$$E(X_x(T\wedge\tau_N)^2) \;\leq\; x^2 + T. \tag{3.21}$$

The derivation of (3.21) is also very similar to the calculations in Lemma 2.1 of [13] (see the estimate (2.9) in [13]) and we omit the details.

Now, (3.20) combined with (3.21) yields

$$0 \leq E\left[X_x(T\wedge\tau_N)e^{-\delta(T\wedge\tau_N)}\right] \leq \left[x^2 + T\right]^{\frac{1}{2}}\left[E(e^{-2\delta(T\wedge\tau_N)})\right]^{\frac{1}{2}}.$$

Combining this with (3.18) and (3.19), we obtain

$$\begin{aligned} &E\left[|\mathcal{V}(0)|e^{-\delta(T\wedge\tau_N)}\right] + p\sqrt{x^2+T}\left[E(e^{-2\delta(T\wedge\tau_N)})\right]^{\frac{1}{2}} \\ &\qquad\qquad + E\int_0^{T\wedge\tau_N} e^{-\delta s}\left[(\beta\gamma X_x(s-) + C(u(s)))ds + p\,dU(s)\right] \;\geq\; \mathcal{V}(x). \end{aligned}$$

Next, first letting N go to infinity, and then taking limit as $T\to\infty$, we obtain

$$\widetilde{J}_p(x,u,U) = E\int_0^{\infty} e^{-\delta s}\left[(\beta\gamma X_x(s-) + C(u(s)))ds + p\,dU(s)\right] \;\geq\; \mathcal{V}(x),$$

where $\widetilde{J}_p$ is as defined in (3.4). Taking the infimum over all admissible policies (X_x, u, U), we get

$$\mathrm{V}_p(x) \geq \mathcal{V}(x), \text{ for all } x \geq 0.$$

This completes the proof. ■

3.2 An optimal control policy

First we describe our candidate for an optimal control policy for the BCP in detail and then prove its optimality in the next theorem (Theorem 3.7). The constant p_0 defined in (3.6) turned out to be a threshold point for the suggested optimal strategy in the following sense: When $0 < p < p_0$, the state space of the optimal state process is a finite interval (after a possible initial jump). When $p \geq p_0$, optimal strategy does not allow any rejections (i.e $U^* \equiv 0$). Thus the state process is independent of p and the state space is the infinite interval $[0, \infty)$. Furthermore, when $p \geq p_0$, the value function $\mathrm{V}_p(\cdot)$ satisfies $\mathrm{V}_p(x) = \mathrm{V}_{p_0}(x)$ for all x. Now we describe our candidate policy which is shown to be optimal in Theorem 3.7.

Definition 3.6 (Optimal policy) *For $0 < p < p_0$, the optimal state process $X^*_{p,x}(\cdot)$ is a reflecting diffusion process on $[0, b^*_p]$ for some $b^*_p > 0$ (as in Lemma 3.10) and it satisfies*

$$X^*_{p,x}(t) = x - \int_0^t u^*_p(X^*_{p,x}(s))ds - \gamma \int_0^t X^*_{p,x}(s)ds + W(t) + L^*_p(t) - U^*_p(t). \tag{3.22}$$

Here $L^*_p(\cdot)$ is the local-time process of $X^*_{p,x}(\cdot)$ at the origin. The feedback-type optimal drift control is given by $u^*_p(X^*_{p,x}(\cdot))$ where $u^*_p(\cdot)$ is a Lipschitz continuous function described in (3.31). Without any ambiguity, we refer to this feedback-type drift control by $u^*_p(\cdot)$. The optimal rejection policy $U^*_p(\cdot)$ satisfies $U^*_p(t) = (x - b^*_p)^+ + U^*_{b^*_p}(t)$ for all $t \geq 0$, where $U^*_{b^*_p}(\cdot)$ is the local time process of $X^*_{p,x}(\cdot)$ at $b^*_p > 0$. Note that $X^*_{p,x}(\cdot)$ makes an initial jump to b^*_p if $x > b^*_p$. We simply identify this policy by $(X^*_{p,x}, u^*_p, U^*_p)$ for $0 < p < p_0$.

For $p \geq p_0$, the same admissible control is optimal for all the values of p and hence $\mathrm{V}_p(x) = \mathrm{V}_{p_0}(x)$, for all $x \geq 0$. Thus, we denote the optimal state process by $X^*_x(\cdot)$ and it is a reflecting diffusion on $[0, \infty)$ which satisfies

$$X^*_x(t) = x - \int_0^t u^*_p(X^*_x(s))ds - \gamma \int_0^t X^*_x(s)ds + W(t) + L^*_p(t), \tag{3.23}$$

with the same notation for the processes as in (3.22). The feedback-type optimal drift is given by $u^*_{p_0}(X^*_x(\cdot))$ where $u^*_{p_0}(\cdot)$ is a Lipschitz continuous function described in (3.37). Hence for all $p \geq p_0$, we take $u^*_p = u^*_{p_0}$ for the optimal drift function. In this case, the optimal rejection process is identically zero and hence X^*_x corresponds to a queue-length process with infinite buffer capacity. Accordingly, we denote this policy by $(X^*_x, u^*_p, 0)$.

Now we state the main theorem of this section.

Theorem 3.7 *(a) For each $p > 0$, the value function $\mathrm{V}_p(\cdot)$ is a convex $\mathcal{C}^2$-function which satisfies the HJB equation in (3.13) together with (3.14). When $p \geq p_0$, $\mathrm{V}_p(x) = \mathrm{V}_{p_0}(x)$ for all $x \geq 0$. Furthermore the feedback-type optimal drift $u^*_p(\cdot)$ in (3.22) and (3.23) satisfies the condition*

$$u^*_p(x) = \Psi(\mathrm{V}'_p(x)), \text{ for all } x \geq 0 \text{ and for each } p > 0, \tag{3.24}$$

where Ψ is as given in (3.9).

(b) When $0 < p < p_0$, the policy $(X^*_{p,x}, u^*_{p,x}, U^*_{p,x})$ described in (3.22) is optimal and b^*_p represents the optimal buffer size. It also satisfies

$$b^*_p = \inf\{x > 0 : \mathrm{V}'_p(x) = p\}. \tag{3.25}$$

If $p \geq p_0$, the policy $(X^*_x, u^*_p, 0)$ described in (3.23) is optimal. Here the state process X^*_x corresponds to a infinite buffer capacity.

Remark 3.8 *When $0 < p < p_0$, b^*_p is finite and the value function $\mathrm{V}_p(\cdot)$ also satisfies $\mathrm{V}''_p(b^*_p) = 0$. In this case, from our optimal policy it follows that $V_p(x) = V_p(b^*_p) + p\ \ (x - b^*_p)$ when $x > b^*_p$.*

Proof. First we consider $0 < p < p_0$. We assume that there exists a point $b^*_p > 0$ and an increasing function $\mathcal{Y}_p$ such that

$$\frac{1}{2}\mathcal{Y}'_p(x) - \Phi(\mathcal{Y}_p(x)) - \gamma x \mathcal{Y}_p(x) + \beta\gamma x = \frac{1}{2}\mathcal{Y}'_p(0) + \delta \int_0^x \mathcal{Y}_p(u)du, \tag{3.26}$$

together with the boundary conditions

$$\mathcal{Y}_p(0) = 0,\ \mathcal{Y}_p(b^*_p) = p,\ \ \mathcal{Y}'_p(b^*_p) = 0,\ \text{and } 0 \leq \mathcal{Y}_p(x) < p \text{ when } 0 \leq x < b^*_p. \tag{3.27}$$

We will verify the existence of such a $b^*_p > 0$ and the function $\mathcal{Y}_p$ in Theorem 3.10. Next introduce

$$\mathcal{V}_p(x) = \begin{cases} \frac{1}{2\delta}\mathcal{Y}'_p(0) + \int_0^x \mathcal{Y}_p(u)du & \text{for all } \ 0 \leq x \leq b^*_p, \\ \\ \mathcal{V}_p(b^*_p) + p(x - b^*_p) & \text{for all } \ x > b^*_p. \end{cases} \tag{3.28}$$

Since $\mathcal{Y}_p(\cdot)$ is an increasing $\mathcal{C}^1$-function on $[0, b^*_p]$, $\mathcal{V}_p(\cdot)$ is a convex $\mathcal{C}^2$-function on $[0, \infty)$. Furthermore, $\mathcal{V}_p(\cdot)$ satisfies

$$\frac{1}{2}\mathcal{V}''_p(x) - \Phi(\mathcal{V}'_p(x)) - \gamma x \mathcal{V}'_p(x) - \delta\mathcal{V}_p(x) + \beta\gamma x = 0 \text{ for } 0 \leq x \leq b^*_p. \tag{3.29}$$

Evaluating (3.29) at $x = b^*_p$ and using (3.27) we obtain

$$\delta\ \mathcal{V}_p(b^*_p) = \beta\gamma b^*_p - p\gamma b^*_p - \Phi(p).$$

A direct computation using this identity and the fact that $p < p_0$ yields

$$\frac{1}{2}\mathcal{V}''_p(x) - \Phi(\mathcal{V}'_p(x)) - \gamma x \mathcal{V}'_p(x) - \delta\mathcal{V}_p(x) + \beta\gamma x > 0 \text{ for } x > b^*_p. \tag{3.30}$$

Hence, (3.27), (3.29) and (3.30) implies that $\mathcal{V}_p$ satisfies all the assumptions of the verification lemma (Lemma 3.4). Therefore, we conclude that $\mathrm{V}_p(x) \geq \mathcal{V}_p(x)$ for all $x \geq 0$. To show that $\mathrm{V}_p(x)$ is indeed equal to $\mathcal{V}_p(x)$ for all $x \geq 0$, we verify that the proposed policy $(X^*_{p,x}, u^*_p, U^*_p)$ in (3.22) (with appropriately defined $u^*_p(\cdot)$) is an admissible policy and the cost $\widetilde{J}_p(x, u^*_p, U^*_p)$ from this policy (as defined in (3.4)) is equal to $\mathcal{V}_p(x)$ for each $x \geq 0$. Thus, it will follow that $\mathrm{V}_p(x) \leq \mathcal{V}_p(x)$ and consequently, $\mathrm{V}_p(x) = \mathcal{V}_p(x)$ for all $x \geq 0$.

For each $0 < p < p_0$, introduce

$$u^*_p(x) = \Psi(\mathcal{V}'_p(x)), \ \ \text{for all } x \geq 0, \tag{3.31}$$

where $\Psi(\cdot)$ is given in (3.9). By (3.28) and (3.31), $u_p^*(\cdot)$ is a Lipschitz continuous function. Thus, $u_p^*(\cdot)$ takes values in $[0, \theta_p]$ where $C'(\theta_p) = p$. This interval $[0, \theta_p]$ is contained in the control set $\mathcal{A}$ by the assumption (3.7). Let $b_p^* > 0$ be as in (3.26) and (3.27). We consider the policy $(X_{p,x}^*, u_p^*, U_p^*)$ with $u_p^*(\cdot)$ defined in (3.31). Since, $u_p^*(\cdot)$ is a Lipschitz continuous function and $X_{p,x}^*$ is a reflecting diffusion on $[0, b_p^*]$, it is evident that $(X_{p,x}^*, u_p^*, U_p^*)$ is an admissible policy. Note that if $x > b_p^*$, the state process makes an initial jump to b_p^* as explained in the discussion below (3.22). For simplicity, we consider $X_{p,x}^*(0) = x$ is in $[0, b_p^*]$, and apply Itô's Lemma to $\mathcal{V}_p(X_{p,x}^*(T))e^{-\delta T}$. We use (3.10) and (3.29), $\mathcal{V}_p'(0) = 0$ and $\mathcal{V}_p'(b_p^*) = p$ to obtain

$$E[\mathcal{V}_p(X_{p,x}^*(T))e^{-\delta T}] = \mathcal{V}_p(x) + E\int_0^T e^{-\delta s}[\beta\gamma X_{p,x}^*(s) + C(u_p^*(X_{p,x}^*(s)))]ds + pE\int_0^T e^{-\delta s} dU_p^*(s).$$

Here $U_p^*(\cdot)$ is the local-time process of $X_{p,x}^*(\cdot)$ at $b_p^* > 0$. Since $\mathcal{V}_p$ is bounded on $[0, b_p^*]$, by letting $T \to \infty$, we obtain

$$\mathcal{V}_p(x) = \widetilde{J}_p(x, u_p^*, U_p^*), \tag{3.32}$$

where $\widetilde{J}_p(\cdot)$ is as given in (3.4). When $x > b_p^*$, there is an initial jump to b_p^* using the rejection process U_p^*. Hence,

$$\widetilde{J}_p(x, u_p^*, U_p^*) = p(x - b_p^*) + \widetilde{J}_p(b_p^*, u_p^*, U_p^*) = p(x - b_p^*) + \mathcal{V}_p(b_p^*) = \mathcal{V}_p(x), \tag{3.33}$$

by (3.28). Hence we have $\mathrm{V}_p(x) \leq \mathcal{V}_p(x)$ (which implies that $\mathrm{V}_p(x) = \mathcal{V}_p(x)$) and therefore, $(X_{p,x}^*, u_p^*, U_p^*)$ is an optimal policy for $0 < p < p_0$. The conclusions (3.24) and (3.25) both follow directly from (3.27) and (3.31). This completes the proof of both parts of the Theorem (3.7), when $0 < p < p_0$.

To prove the theorem for $p \geq p_0$, we assume the existence of an increasing function $\mathcal{Y}_0$ which satisfies

$$\frac{1}{2}\mathcal{Y}_0'(x) - \Phi(\mathcal{Y}_0(x)) - \gamma x \mathcal{Y}_0(x) + \beta\gamma x = \frac{1}{2}\mathcal{Y}_0'(0) + \delta\int_0^x \mathcal{Y}_0(u)du, \quad \text{for all} \quad x \geq 0, \tag{3.34}$$

together with the boundary conditions

$$\mathcal{Y}_0(0) = 0, \; 0 \leq \mathcal{Y}_0(x) < p_0 \text{ for all } x \geq 0 \; \text{ and } \lim_{x\to\infty} \mathcal{Y}_0(x) = p_0. \tag{3.35}$$

We will also verify the existence of such a function $\mathcal{Y}_0$ in Theorem 3.10. Introduce

$$\mathcal{V}_0(x) = \frac{1}{2\delta}\mathcal{Y}_0'(0) + \int_0^x \mathcal{Y}_0(u)du \text{ for all } x \geq 0. \tag{3.36}$$

Since $\mathcal{Y}_0(\cdot)$ is an increasing $\mathcal{C}^1$-function, $\mathcal{V}_p(\cdot)$ is a convex $\mathcal{C}^2$-function. We take any $p \geq p_0$. Then a direct computation using (3.34) and (3.35) verifies that $\mathcal{V}_0$ satisfies all the assumptions of the verification lemma (Lemma 3.4). Hence, we obtain $\mathrm{V}_p(x) \geq \mathcal{V}_0(x)$ for all $x \geq 0$. Now we prove $\mathrm{V}_p(x) = \mathcal{V}_0(x)$ for all $x \geq 0$. For each $p \geq p_0$, we introduce

$$u_p^*(x) = \Psi(\mathcal{V}_0'(x)), \quad \text{for all } x \geq 0, \tag{3.37}$$

where $\Psi(\cdot)$ is given in (3.9). Notice that for $p \geq p_0$, $u_p^*(x) = u_{p_0}^*(x)$ for all $x \geq 0$, since $\mathcal{V}_0$ defined in (3.36) depends only on p_0. We intend to show that $(X_x^*, u_p^*, 0)$ is an admissible policy for all $p \geq p_0$, and $\widetilde{J}_p(x, u_p^*, 0) = \mathcal{V}_0(x)$ for all $x \geq 0$. Note that $u_p^*(\cdot)$ is a Lipschitz continuous function and $X^*_{p,x}$ is a reflecting diffusion on $[0,+\infty]$ with a reflecting barrier at the origin. By (3.9) and (3.37), $u^*_p(\cdot)$ take values in $[0,\theta_0]$ where $C'(\theta_0)=p_0$. Notice that $[0,\theta_0]$ is contained in the control set $\mathcal{A}$ by (3.7). Therefore, $(X^*_x, u^*_p, 0)$ is an admissible policy.

Now X^*_x satisfies (3.23) with optimal drift $u^*_p(\cdot)$ defined in (3.37). Hence we apply Itô's Lemma to $\mathcal{V}_0(X^*_x(T))e^{-\delta T}$ to obtain

$$E[\mathcal{V}_0(X^*_x(T))e^{-\delta T}] = \mathcal{V}_0(x) - E\int_0^T e^{-\delta s}[\beta\gamma X^*_x(s) + C(u^*_p(X^*_x(s)))]ds.$$

To verify $\lim_{T\to\infty} E[\mathcal{V}_0(X^*_x(T))e^{-\delta T}] = 0$, by (3.35), it suffices to show that

$$\lim_{T\to\infty} E[X^*_x(T)e^{-\delta T}] = 0.$$

For this, we again apply Itô's Lemma to $[X^*_x(T)]^2$, using (3.23) and eliminate the negative terms to get the estimate $E[X^*_x(T)]^2 \le x^2 + T$. This yields $\lim_{T\to\infty} E[X^*_x(T)e^{-\delta T}] = 0$. Hence, using a similar approach as used in deriving (3.32), we obtain

$$\mathcal{V}_0(x) = \widetilde{J}_p(X^*_x, u^*_p, 0), \quad \text{for all } x \ge 0,\ p \ge p_0,$$

and $(X^*_x, u^*_p, 0)$ is an optimal policy for each $p \ge p_0$. Furthermore, the feedback-type drift control u^*_p is given by $u^*_p(x) \equiv u^*_{p_0}(x) = \Psi(\mathcal{V}'_0(x)) = \Psi(\mathrm{V}'_{p_0}(x))$, for all $x \ge 0$. Since, $\mathcal{V}_0(\cdot)$ is a $\mathcal{C}^2$-function, the proof of Theorem 3.7 for the case $p \ge p_0$ is also complete. ■

Remark 3.9 *The above proof shows that there exists a real number b^*_p (b^*_p is considered as $+\infty$ in the case of $p \ge p_0$) and a $\mathcal{C}^2$-function $\mathcal{V}_p$ which satisfies*

$$\frac{1}{2}\mathcal{V}''_p(x) - \Phi(\mathcal{V}'_p(x)) - \gamma x\mathcal{V}'_p(x) - \delta\mathcal{V}_p(x) + \beta\gamma x = 0 \text{ for } 0 \le x \le b^*_p, \tag{3.38}$$

$$\mathcal{V}'_p(0) = 0 \quad \text{and } \mathcal{V}'_p(x) = p, \text{ for } x \ge b^*_p. \tag{3.39}$$

Since, $\mathcal{V}_p(x) \equiv \mathrm{V}_p(x)$ for all $x \ge 0$, where $\mathrm{V}_p(\cdot)$ is the value function defined in (3.5), the pair $(b^*_p, \mathcal{V}_p(\cdot))$ is unique.

It remains to verify the existence of a function $\mathcal{Y}_p(\cdot)$ which satisfies (3.26) and (3.27) and a function $\mathcal{Y}_0(\cdot)$ which satisfies (3.34) and (3.35). We address this issue in the next subsection.

3.3 A parametrization method

Our aim here is to establish the existence of a solution $\mathcal{Y}_p(\cdot)$ which satisfies (3.26)-(3.27) and another solution $\mathcal{Y}_0(\cdot)$ which satisfies (3.34) and (3.35). This will be achieved in the following theorem and it will complete the proof of the Theorem 3.7.

Theorem 3.10 *(i) For each p in $(0,p_0)$, there exists a point $b^*_p > 0$ and an increasing function $\mathcal{Y}_p(\cdot)$ which satisfies (3.26) and (3.27).*

(ii) There also exists an increasing function $\mathcal{Y}_0(\cdot)$ defined on $[0,\infty)$, which satisfies (3.34) and (3.35).

The proof of this theorem will be given at the end of this section, since it needs several results about the behavior of a parametric family of solutions to the differential equation in (3.41) below.

First we extend the function Φ defined in (3.8)-(3.10) to negative real axis by setting

$$\Phi(y) = 0, \text{ for all } y \leq 0. \tag{3.40}$$

Then, by the assumptions on the cost function C (Assumption 2.4), (3.11) and (3.12), it is clear that Φ' is a Lipschitz continuous function on $I\!R$. For our purposes, only the behavior of Φ on the interval $[0, p_0]$ is crucial.

Next we consider the following parametric family of differential equations:

$$\begin{cases} \mathcal{Y}_r'(x) - 2\Phi(\mathcal{Y}_r(x)) - 2\gamma x \mathcal{Y}_r(x) + 2\beta\gamma x = r + 2\delta \int_0^x \mathcal{Y}_r(u) du \\ \mathcal{Y}_r(0) = 0, \ \mathcal{Y}_r'(0) = r. \end{cases} \tag{3.41}$$

We differentiate the above equation and use (3.12) to obtain

$$\mathcal{Y}_r''(x) - 2\Psi(\mathcal{Y}_r(x))\mathcal{Y}_r'(x) - 2\gamma x \mathcal{Y}_r'(x) - 2(\gamma + \delta)\mathcal{Y}_r(x) + 2\beta\gamma = 0. \tag{3.42}$$

Since Ψ is a $\mathcal{C}^1$-function, this second order non-linear differential equation with the initial data $\mathcal{Y}_r(0) = 0$ and $\mathcal{Y}_r'(0) = r$ has a unique solution which is valid on the interval $[0, \omega_r)$ where ω_r is the explosion point for $\mathcal{Y}_r$ (See [16]), and $0 < \omega_r \leq +\infty$. Consequently, (3.41) has a unique solution $\mathcal{Y}_r$ which is valid on $[0, \omega_r)$. Furthermore, this solution $\mathcal{Y}_r(x)$ is jointly continuous in (r, x). (See chapter 5 of [16]) and we will use this fact in our analysis of (3.41).

Our next proposition describes the properties of the solution $\mathcal{Y}_r$.

Proposition 3.11 *For the family of solutions $(\mathcal{Y}_r(\cdot))_{r>0}$, the following properties hold:*

(i) *if $r_1 > r_2 > 0$ then $\mathcal{Y}_{r_1}(x) > \mathcal{Y}_{r_2}(x)$ for all $0 < x < \omega_{r_1} \wedge \omega_{r_2}$.*
Furthermore, $\mathcal{Y}_{r_1}(x) > (r_1 - r_2)x + \mathcal{Y}_{r_1}(x)$ on this interval $(0, \omega_{r_1} \wedge \omega_{r_2})$.

(ii) *If $\mathcal{Y}_r'(\xi) = 0$ for some $\xi > 0$, then $\mathcal{Y}_r(\xi) \neq p_0$ where $p_0 > 0$ is given in (3.6). Furthermore, if $x = \xi > 0$ is the local maximum for $\mathcal{Y}_r$ then $\mathcal{Y}_r(\xi) < p_0$. Also, $\mathcal{Y}_r$ cannot have any local minima.*

(iii) *There exist $r_0 > 0$ such that for each $r > r_0$, $\mathcal{Y}_r$ does not have any local maxima and $\mathcal{Y}_r(x)$ is increasing to ∞ as x increases to ω_r.*

(iv) *For each $r > 0$, $\mathcal{Y}_r$ has a positive local maximum on $(0, \infty)$ if and only if $\mathcal{Y}_r(z) = 0$ for some $z > 0$.*

Proof. Let $r_1 > r_2 > 0$. since $\mathcal{Y}_{r_1}(0) = \mathcal{Y}_{r_2}(0) = 0$ and $\mathcal{Y}_{r_1}'(0) = r_1 > r_2 = \mathcal{Y}_{r_2}'(0)$, it follows that $\mathcal{Y}_{r_1}(x) > \mathcal{Y}_{r_2}(x)$ for all x in an interval $(0, \delta)$ for some $\delta > 0$. Now suppose $\mathcal{Y}_{r_2}(z) \geq \mathcal{Y}_{r_1}(z)$ for some $z \geq 0$, then there is a point $c \geq \delta > 0$ such that $\mathcal{Y}_{r_2}(c) = \mathcal{Y}_{r_1}(c)$ and $\mathcal{Y}_{r_2}(x) < \mathcal{Y}_{r_1}(x)$ when $0 < x < c$. Then using (3.41),

$$\begin{aligned} &\mathcal{Y}_{r_1}'(x) - \mathcal{Y}_{r_2}'(x) \\ &\qquad = (r_1 - r_2) + 2\gamma x(\mathcal{Y}_{r_1}(x) - \mathcal{Y}_{r_2}(x)) + 2\left(\Phi(\mathcal{Y}_{r_1}(x)) - \Phi(\mathcal{Y}_{r_2}(x))\right) + 2\delta \int_0^x [\mathcal{Y}_{r_1}(u) - \mathcal{Y}_{r_2}(u)] du. \end{aligned}$$

Since Φ is an increasing function, this implies that $\mathcal{Y}'_{r_1}(x) - \mathcal{Y}'_{r_2}(x) > (r_1 - r_2)$ for each x in $(0, c)$. Hence, $\mathcal{Y}_{r_1}(c) = \mathcal{Y}_{r_2}(c)$ is impossible and the same argument implies that $\mathcal{Y}'_{r_1}(x) - \mathcal{Y}'_{r_2}(x) > (r_1 - r_2)$ for all x in $(0, \omega_{r_1} \wedge \omega_{r_2})$. Consequently $\mathcal{Y}_{r_1}(x) > (r_1 - r_2)x + \mathcal{Y}_{r_1}(x)$ on this interval $(0, \omega_{r_1} \wedge \omega_{r_2})$. This completes the proof of part (i).

For part (ii), let $\xi > 0$ be a point which satisfies $\mathcal{Y}'_r(\xi) = 0$. Suppose that $\mathcal{Y}_r(\xi) = p_0$ where p_0 is given in (3.37). Now let

$$x_0 = \inf\{\xi > 0 : \mathcal{Y}_r(\xi) = p_0 \text{ and } \mathcal{Y}'_r(\xi) = 0\}.$$

Then $x_0 > 0$, $\mathcal{Y}_r(x_0) = p_0$ and $\mathcal{Y}'_r(x_0) = 0$. The function $\mathcal{Y}_r$ also satisfies (3.42) with the same initial data $\mathcal{Y}(x_0) = p_0$ and $\mathcal{Y}'(x_0) = 0$. Since Ψ is a $\mathcal{C}^1$-function, this initial value problem has a unique solution in an interval $(x_0 - \delta,\ x_0 + \delta)$ for some $\delta > 0$ where $x_0 > \delta$. Hence, $\mathcal{Y}_r(x) \equiv p_0$ on $(x_0 - \delta,\ x_0 + \delta)$ and this contradicts with the definition of x_0. Consequently $\mathcal{Y}_r(\xi) \neq p_0$ if $\mathcal{Y}'_r(\xi) = 0$.

Next, if $\mathcal{Y}'_r(\xi) = 0$, by (3.42) we obtain,

$$\frac{1}{2}\mathcal{Y}''_r(\xi) = (\delta + \gamma)(\mathcal{Y}_r(\xi) - p_0). \tag{3.43}$$

Hence if $x = \xi$ is a local maximum, then $\mathcal{Y}''_r(\xi) \leq 0$ and by (3.43) we obtain $\mathcal{Y}_r(\xi) \leq p_0$. Since $\mathcal{Y}'_r(\xi) = 0$, we know that $\mathcal{Y}_r(\xi) \neq p_0$ and consequently, $\mathcal{Y}_r(\xi) < p_0$. If $x = \xi > 0$ is a local minimum then $\mathcal{Y}'_r(\xi) = 0$ and $\mathcal{Y}''_r(\xi) \geq 0$. Then by (3.43), $\mathcal{Y}_r(\xi) \geq p_0$. Since $\mathcal{Y}_r(0) = 0$ and $\mathcal{Y}'_r(0) = r > 0$, it follows that $\mathcal{Y}_r$ is strictly increasing in an interval $(0, \delta)$ for some $\delta > 0$. These two facts imply the existence of a local maximum at $x = z$ where $0 < z < \xi$ and $\mathcal{Y}_r(z) > p_0$. This is a contradiction. Hence $\mathcal{Y}_r$ cannot have any local minima. This completes the proof of part (ii).

To prove part (iii), we pick $r_1 > 0$, then by the initial conditions in (3.41), $\mathcal{Y}_{r_1}(x) > 0$ for all x in $(0, 2\delta_{r_1})$ for some $\delta_{r_1} > 0$. For $r > r_1$, using (3.41) and part (i) of this proposition, we obtain

$$\mathcal{Y}'_r(x) > r - 2\beta\gamma x \quad \text{for} \quad 0 < x < 2\delta_{r_1}.$$

Next, we pick $r_0 > r_1$ such that $(r_0 - 2\beta\gamma\delta_{r_1})\delta_{r_1} > p_0$. Hence $\mathcal{Y}'_{r_0}(x) > r_0 - 2\beta\gamma\delta_{r_1}$ when $0 < x < \delta_{r_1}$ and consequently for $r > r_0$, $\mathcal{Y}_r(\delta_{r_1}) > \mathcal{Y}_{r_1}(\delta_{r_1}) \geq (r_0 - 2\beta\gamma\delta_{r_1})\delta_{r_1} > p_0 > 0$. By part (ii), $\mathcal{Y}_r$ cannot have any local maxima when $\mathcal{Y}_r(x) > p_0$ and therefore, we conclude that $\mathcal{Y}_r(\cdot)$ is an increasing function when $x > \delta_{r_1}$.

Now if $\lim_{x\to\omega_r} \mathcal{Y}_r(x) = \lambda_0$ exists and if λ_0 is finite, by integrating (3.41), it is easy to observe that ω_r is infinite. Then again using (3.41), we obtain

$$\lim_{x\to\infty} \frac{\mathcal{Y}'_r(x)}{x} = 2(\delta + \gamma)(\lambda_0 - p_0). \tag{3.44}$$

Clearly $\lambda_0 > p_0$, thus the above limit is positive and $\lim_{x\to\infty} \mathcal{Y}_r(x) = +\infty$. This is a contradiction and hence $\lambda_0 = +\infty$. Thus $\mathcal{Y}_r$ is increasing to $+\infty$ as x increases to ω_r. This completes part (iii).

Now let $x = \xi > 0$ be the first local maximum of $\mathcal{Y}_r$ on $(0, +\infty)$. Then $p_0 > \mathcal{Y}_r(\xi) > 0$, $\mathcal{Y}'_r(\xi) = 0$ and $0 < \mathcal{Y}_r(x) < \mathcal{Y}_r(\xi)$ when $0 < x < \xi$. By (3.43), $\mathcal{Y}''_r(\xi) < 0$ and by part (ii), $\mathcal{Y}_r$ does not have any local minima. Therefore $\mathcal{Y}_r$ is decreasing when $x > \xi$. If $\lim_{x\to\omega_r} \mathcal{Y}_r(x)$ is finite, then we can use (3.44) and the argument above to conclude $\omega_r \equiv +\infty$ and $\lim_{x\to\infty} \frac{\mathcal{Y}'_r(x)}{x} < 0$. Thus

$\lim_{x\to\infty}\mathcal{Y}_r(x) = -\infty$ and this is a contradiction. Hence $\lim_{x\to\omega_r}\mathcal{Y}_r(x) = -\infty$ and as a consequence, $\mathcal{Y}_r(z) = 0$ for some $z > \xi$.

Conversely, if $\mathcal{Y}_r(z) = 0$ for $z > 0$, since $\mathcal{Y}_r(0) = 0$ and $\mathcal{Y}_r'(0) = r > 0$ it is clear that there is a local maximum at a point $\xi > 0$ where $0 < \xi < z$ and $\mathcal{Y}_r(\xi) > 0$. This completes the proof of the proposition. ■

Remark 3.12 *One reason that the value* $p_0 = \frac{\beta\gamma}{(\delta+\gamma)}$ *is a critical value in the analysis of the parametric family of solutions to (3.41) is that the constant function* $\mathcal{Y}(x) = p_0$ *is the only constant solution to (3.42). But note that, it does not satisfy (3.41).*

Proposition 3.13 *There exists* $\hat{r} > 0$ *which satisfies the following conditions:*

(i) If $0 < r < \hat{r}$ *then there exists* $z_r > 0$ *such that* $\mathcal{Y}_r(z_r) = 0$ *and the set* $\{x > 0 : \mathcal{Y}_r(x) > 0\}$ *is equal to the open interval* $(0, z_r)$*. Furthermore, let*

$$H(r) = \max_{x>0} \mathcal{Y}_r(x). \tag{3.45}$$

Then $H(r)$ *is finite,* $H(r) = \max_{0<x<z_r}\mathcal{Y}_r(x)$ *and* $0 < H(r) < p_0$*.*

(ii) When $r = \hat{r}$*,* $\mathcal{Y}_{\hat{r}}$ *is strictly increasing,* $\omega_{\hat{r}} \equiv +\infty$ *and* $\lim_{x\to+\infty}\mathcal{Y}_{\hat{r}}(x) = p_0$*.*

(iii) If $r > \hat{r}$*,* $\mathcal{Y}_r$ *increases to* $+\infty$ *when* x *increases to* ω_r*.*

Proof. First we consider the solution $\mathcal{Y}_0$ to (3.41) which corresponds to $r = 0$. Using (3.42), the fact that $\Psi(0) = 0$, and the initial conditions $\mathcal{Y}_0(0) = \mathcal{Y}_0'(0) = 0$, we obtain $\mathcal{Y}_0''(0) = -2\beta\gamma < 0$. Hence, there exists an $\epsilon_0 > 0$ such that $\mathcal{Y}_0$ is strictly concave on $(-2\epsilon_0, 2\epsilon_0)$ and $\mathcal{Y}_0$ has a local maximum at $x = 0$. Consequently, $\mathcal{Y}_0(\epsilon_0) < 0$. Since $\mathcal{Y}_r(x)$ is jointly continuous in (r, x) and using part (i) of Proposition 3.11, we can find $\eta_0 > 0$ such that $\mathcal{Y}_r(\epsilon_0) < 0$ for all $0 \le r < \eta_0$. Thus, for each such r in $(0, \eta_0)$, $\mathcal{Y}_r$ has a positive local maximum ξ_r in $(0, \epsilon_0)$ and a zero at z_r in $(0, \epsilon_0)$ where $0 < \xi_r < z_r < \epsilon_0$.

Introduce

$$\hat{r} = \sup\{r > 0 : \mathcal{Y}_r(x) = 0 \text{ for some } x > 0\}. \tag{3.46}$$

The interval $(0, \eta_0)$ is in the above set and thus $\hat{r}$ is well defined. Let r_0 be as in part (iii) of the Proposition 3.11. Then clearly $\hat{r} \le r_0$. Consequently $0 < \eta_0 \le \hat{r} \le r_0 < +\infty$. Next, by parts (i) and (iv) of Proposition 3.11, it clearly follows that for each $0 < r < \hat{r}$, $\mathcal{Y}_r(x) = 0$ for some $x > 0$. We let

$$z_r = \inf\{x > 0 : \mathcal{Y}_r(x) = 0\}.$$

By part (ii) of Proposition 3.11, each $\mathcal{Y}_r$ can have at most one local maximum and then we can deduce that $H(r)$ is finite, $H(r) = \max_{0<x<z_r}\mathcal{Y}_r(x)$ and $0 < H(r) < p_0$. This completes part (i).

Since $0 < H(r) < p_0$ for each $r < \hat{r}$ and $\mathcal{Y}_r(x)$ is jointly continuous in (r, x), it follows that $0 < \mathcal{Y}_{\hat{r}}(x) < p_0$, for all $x \in (0, \omega_{\hat{r}})$. Suppose that there is a $\xi > 0$ with $\mathcal{Y}_{\hat{r}}'(\xi) = 0$, then $\mathcal{Y}_{\hat{r}}(\xi) < p_0$ by part (ii) of Proposition 3.11. Now using (3.43), we have $\mathcal{Y}_r''(\xi) < 0$ and $x = \xi$ is a strict local maximum for $\mathcal{Y}_r$. Therefore we can employ the joint continuity of $\mathcal{Y}_r(x)$ in (r, x) and the monotonicity of $\mathcal{Y}_r$ in r as in part (i) of the Proposition 3.11 to conclude that for some $r > \hat{r}$, $\mathcal{Y}_r$

also has a local maximum in a neighborhood of ξ when $|r-\hat{r}|$ is sufficiently small. This contradicts with the definition of $\hat{r}$ in (3.46). Hence $\mathcal{Y}_{\hat{r}}'(x) \neq 0$ for all $x \geq 0$ and $\mathcal{Y}_{\hat{r}}$ is a C^2-function. But, $\mathcal{Y}_{\hat{r}}'(0) = \hat{r} > 0$ and consequently $\mathcal{Y}_{\hat{r}}'(x) > 0$ for all $0 < x < \omega_{\hat{r}}$. Thus $\mathcal{Y}_{\hat{r}}$ is an increasing function which satisfies $0 < \mathcal{Y}_{\hat{r}}(x) \leq p_0$ and (3.41). If $\mathcal{Y}_{\hat{r}}(x_1) = p_0$ for some x_1, then it is a local maximum and $\mathcal{Y}_{\hat{r}}'(x_1) = 0$. Then by the uniqueness of the solutions to the differential equation (3.42), it follows that $\mathcal{Y}_{\hat{r}}(x) = p_0$ for all x which is a contradiction. Hence $0 < \mathcal{Y}_{\hat{r}}(x) < p_0$ for all x. By integrating (3.41) it is evident that $\mathcal{Y}_{\hat{r}}(x)$ is finite for each x and thus $\omega_{\hat{r}}(x)$ is finite for each x and thus $\omega_{\hat{r}} \equiv +\infty$. Now let $\lambda_0 = \lim_{x\to\infty} \mathcal{Y}_{\hat{r}}(x)$. Then $0 < \lambda_0 \leq p_0$. By (3.44), $\lim_{x\to\infty} \frac{\mathcal{Y}_{\hat{r}}'(x)}{x} = 2(\delta+\gamma)(\lambda_0 - p_0)$. Since $\mathcal{Y}_{\hat{r}}'(x) > 0$ for all x, it follows that $\lambda_0 \geq p_0$. Hence $\lambda_0 = p_0$ and thus part (ii) follows.

When $r > \hat{r}$, the definition of $\hat{r}$ and part (iv) of the Proposition 3.11 implies that $\mathcal{Y}_r$ cannot have any local maxima. Also, if $\mathcal{Y}_r'(\xi) = 0$ for some $\xi > 0$, since $\mathcal{Y}_r$ does not have any positive local maxima, equation (3.43) and part (ii) of the Proposition 3.11 implies that $\mathcal{Y}_r''(\xi) > 0$ and hence $x = \xi$ is a strict local minimum. But $\mathcal{Y}_r(0) = 0$ and $\mathcal{Y}_r'(0) = r > 0$, therefore $\mathcal{Y}_r$ must have a positive local maximum at some point in $(0, \xi)$ and this is a contradiction. Consequently, $\mathcal{Y}_r'(x) > 0$ for all $0 < x < \omega_r$. Suppose $\lim_{x\to\omega_r} \mathcal{Y}_r(x)$ is finite, say λ_0, then $0 < \mathcal{Y}_r(x) < \lambda_0$ for all $0 < x < \omega_r$. Thus by integrating (3.41), we obtain $\omega_r = +\infty$ and (3.44) holds. But $\mathcal{Y}_r(x) > (r-\hat{r})x + \mathcal{Y}_{\hat{r}}(x)$ each x, by part (i) of the Proposition 3.11. Consequently $\lim_{x\to\infty} \mathcal{Y}_r(x) = +\infty$ and hence $\lambda_0 = +\infty$ and this is a contradiction.

Therefore, we conclude that $\lim_{x\to\omega_r} \mathcal{Y}_r(x) = +\infty$. This completes the proof. ■

Proposition 3.14 *Let the point $\hat{r}$ and the function H be as in Proposition 3.13. Then*

(i) H is a continuous strictly increasing function defined on $(0, \hat{r})$ and it takes all the values in the interval $(0, p_0)$.

(ii) $\lim_{r\to 0+} H(r) = 0$ and $\lim_{r\to\hat{r}^-} H(r) = p_0$.

Proof. Part (i) of the Proposition 3.13 implies that $H(r)$ is finite and $0 < H(r) < p_0$ for each r in $(0, \hat{r})$. Also there is a point ξ_r such that $0 < \xi_r < z_r$ and $H(r) = \mathcal{Y}_r(\xi_r)$. By part (ii) of the Proposition 3.13, we have $\mathcal{Y}_r(\xi_r) < \mathcal{Y}_{\hat{r}}(\xi_r) < p_0$. Therefore, by (3.43), $\mathcal{Y}_r''(\xi_r) < 0$ and $x = \xi_r$ is a strict local maximum. By part (ii) of Proposition 3.11, $\mathcal{Y}_r$ cannot have any local minima and therefore this local maximum point $x = \xi_r$ is unique. Since $\mathcal{Y}_r(x)$ is jointly continuous in (r, x) and using part (i) of Proposition 3.11, it evidently follows that $H(\cdot)$ is a continuous strictly increasing function on $(0, \hat{r})$. This proves part (i).

When $r = 0$, the function $\mathcal{Y}_0$ has a strict local maximum $x = 0$ and is concave in a neighborhood of $x = 0$ as we have noticed in the proof of part (i) of Proposition 3.13. Thus, we can pick a $\delta_0 > 0$ such that $\mathcal{Y}_0(x) < 0$ on $(0, 2\delta_0)$. In particular, $\mathcal{Y}_0(\delta_0) < 0$. For a given $\epsilon > 0$, using part (i) of Proposition 3.11 and joint continuity of $\mathcal{Y}_r(x)$ in both r and x, we can find $\eta_0 > 0$ such that $\mathcal{Y}_r(\delta_0) < 0$ and $|\mathcal{Y}_0(x) - \mathcal{Y}_r(x)| < \epsilon$ for all x in $[0, \delta_0]$ and for all r in $[0, \eta_0)$. Thus $0 < H(r) < \epsilon$ for each $0 < r < \eta_0$. Consequently $\lim_{r\to 0^+} H(r) = 0$. The fact that $\lim_{r\to\hat{r}} H(r) = p_0$ can also be proved by combining the joint continuity of $\mathcal{Y}_r(x)$, the monotonicity property of $\mathcal{Y}_r$ as in part (i) of the Proposition 3.11 and the fact that $\lim_{x\to\infty} \mathcal{Y}_{\hat{r}}(x) = p_0$. This completes the proof. ■

Figure 2: Nature of the family of solutions $\mathcal{Y}_r$

Proof of Theorem 3.10. Let $0 < p < p_0$. By the previous proposition, there exists a unique r_p in $(0, \hat{r})$ and a unique point ξ_{r_p} such that

$$p = H(r_p) = \mathcal{Y}_{r_p}(\xi_{r_p}).$$

Furthermore $\mathcal{Y}'_{r_p}(x) > 0$ when $0 < x < \xi_{r_p}$. We relabel the point ξ_{r_p} by b^*_p and the function $\mathcal{Y}_{r_p}$ by $\mathcal{Y}_p$ on the interval $[0, b^*_p]$. Then the point $b^*_p > 0$ and the function $\mathcal{Y}_p(\cdot)$ satisfies (3.26) and (3.27).

For part (ii), consider $\hat{r} > 0$ given in (3.46) and the associated function $\mathcal{Y}_{\hat{r}}(\cdot)$ as described in the Proposition 3.13. We simply relabel this function as $\mathcal{Y}_0(\cdot)$. Then clearly $\mathcal{Y}_0$ satisfies (3.34) and (3.35). This completes the proof. ■

4 Asymptotic optimality

In this section we provide the proof of our main result, Theorem 2.6. This proof involves showing the policy proposed in Definition 2.5 is asymptotically optimal, using Theorem 3.7 from Section 3. The proof of Theorem 2.6 and other weak convergence results leading to this proof are given in Subsection 4.2. These proofs also use properties of the "regulator-maps" discussed first in Subsection 4.1.

4.1 Regulator maps

Definition 4.1 (generalized regulator maps) *Let $u : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous, non-negative function and $\gamma > 0$ be a constant.*

One-sided generalized regulator mapping *is a mapping*

$$(\phi^{u,\gamma}, \psi^{u,\gamma}) : \mathcal{D}([0,\infty), I\!R) \to \mathcal{D}([0,\infty), [0,\infty) \times [0,\infty))$$

such that for any given $w \in \mathcal{D}([0,\infty), I\!R)$ *with* $w(0) \geq 0$, $(\tilde{q}, \tilde{\ell}) \equiv (\phi^{u,\gamma}, \psi^{u,\gamma})(w)$ *satisfies*

(i) $\tilde{q}(t) = w(t) - \int_0^t [u(\tilde{q}(s)) + \gamma \tilde{q}(s)] ds + \tilde{\ell}(t) \geq 0, \;\; \forall t \geq 0,$

(ii) $\tilde{\ell}(\cdot)$ *is nondecreasing,* $\tilde{\ell}(0) = 0$ *and* $\int_0^\infty \tilde{q}(t) d\tilde{\ell}(t) = 0.$

Two-sided generalized regulator mapping *is defined for any real* $b \in (0,\infty)$ *as a mapping*

$$(\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma}) : \mathcal{D}([0,\infty), I\!R) \to \mathcal{D}([0,\infty), [0,b] \times [0,\infty) \times (0,\infty))$$

such that for any given $w \in \mathcal{D}([0,\infty), I\!R)$ *with* $0 \leq w(0) \leq b$ *and* $(\tilde{q}, \tilde{\ell}, \tilde{u}) \equiv (\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma})(w)$ *satisfies*

(i) $\tilde{q}(t) = w(t) - \int_0^t [u(\tilde{q}(s)) + \gamma \tilde{q}(s)] ds + \tilde{\ell}(t) - \tilde{u}(t) \in [0,b], \; \forall t \geq 0,$

(ii) $\tilde{\ell}(\cdot), \tilde{u}(\cdot)$ *are both nondecreasing,* $\tilde{\ell}(0) = \tilde{u}(0) = 0,$ $\int_0^\infty \tilde{q}(t) d\tilde{\ell}(t) = \int_0^\infty (b - \tilde{q}(t))^+ d\tilde{u} = 0.$

The argument for the existence and uniqueness of the two types of generalized regulator mappings can be found in Proposition 4 of [32] and Lemma 1 of [26]. Let (ϕ, ψ) be the conventional one-sided regulator map (or the Skorohod map) on $[0,\infty)$ and $(\phi_b, \psi_{1,b}, \psi_{2,b})$ be the two-sided regulator map (or the Skorohod map) on $[0,b]$ (see [28], [17]). The properties of the two sided regulator map described below are generalizations of the work of [32]. Observe that these conventional one-sided and two-sided regulator maps can be obtained from Definition 4.1 by setting $u \equiv 0$ and $\gamma = 0$. As shown in [32], the explicit forms of the generalized regulator mappings can be given in terms of the conventional regulator maps as:

$$\begin{aligned} (\phi^{u,\gamma}, \psi^{u,\gamma})(w) &= (\phi, \psi)(\mathcal{M}^{u,\gamma}(w)), \\ (\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma})(w) &= (\phi_b, \psi_{1,b}, \psi_{2,b})(\mathcal{M}_b^{u,\gamma}(w)), \end{aligned} \tag{4.1}$$

where for a given w as in Definition 4.1, $\mathcal{M}^{u,\gamma}(w), \mathcal{M}_b^{u,\gamma}(w)$ are defined as follows: If $\nu(\cdot) \equiv \mathcal{M}^{u,\gamma}(w(\cdot)), \nu_b(\cdot) \equiv \mathcal{M}_b^{u,\gamma}(w(\cdot))$, then they describe the unique solutions to the following integral equations:

$$\begin{aligned} \nu(t) &= w(t) - \int_0^t [u(\phi(\nu)(s)) + \gamma \phi(\nu)(s)] ds, \; t \geq 0, \\ \nu_b(t) &= w(t) - \int_0^t [u(\phi_b(\nu_b)(s)) + \gamma \phi_b(\nu_b)(s)] ds, t \geq 0. \end{aligned} \tag{4.2}$$

In the following Lemma, we state some standard properties of the conventional regulator maps (Skorohod maps) and outline their proofs. Throughout this section, we will denote $\sup_{0 \leq s \leq T} |z(s)|$ by $||z||_T$, for any $z \in \mathcal{D}([0,\infty), I\!R)$ and $T > 0$.

Lemma 4.2 *Let $\{w_n\}, w$ be as in Definition 4.1 and let $T > 0$ be fixed. Then for some universal constants $c_1 > 0$ and $c_2 > 0$, the following properties hold for the conventional regulator maps:*

$$
\begin{aligned}
&(a) \quad ||\phi(w)||_T \vee ||\psi(w)||_T \le c_1 ||w||_T, \ \ ||\phi_b(w)||_T \vee ||\psi_{1,b}(w)||_T \vee ||\psi_{2,b}(w)||_T \le c_2 ||w||_T \\
&(b) \quad \text{If } \lim_{n\to 0} ||w_n - w||_T = 0, \text{ then } \ \lim_{n\to 0} ||\phi(w_n) - \phi(w)||_T \vee ||\psi(w_n) - \psi(w)||_T = 0, \\
&\qquad \lim_{n\to 0} ||\phi_b(w_n) - \phi_b(w)||_T \vee ||\psi_{1,b}(w_n) - \psi_{1,b}(w)||_T \vee ||\psi_{2,b}(w_n) - \psi_{2,b}(w)||_T = 0.
\end{aligned}
$$

Therefore, these maps are continuous under the metric of uniform convergence on compact sets.

Proof. The proof of the first part of the lemma above is similar to that of the part (iii) of Proposition 4 of [32]. The Lipschitz continuity of the one-sided regulator maps (easy to verify from the definition, see also Section 13.5 of [33]) implies the first inequality in part (a) and the first convergence in part (b). The two sided regulator map ϕ_b is also Lipschitz in this uniform topology (Theorem 14.8.1 of [33]), which proves the claimed properties of ϕ_b in parts (a) and (b) above. Following the arguments in the proof of part (ii) of Proposition 4 of [32], and the fact that $Osc(w,[0,T]) \doteq \sup_{0\le r\le s\le T} |w(r) - w(s)| \le 2||w||_T$, we can obtain

$$||\psi_{1,b}(w)||_T \vee ||\psi_{2,b}(w)||_T \le c_2' ||w||_T,$$

for some constant $c_2' > 0$, which completes the proof of part (a) above. Theorem 14.8.1 of [33] completes the proof of the second part (b). ■

Now using Lemma 4.2, we prove the following specific properties of the generalized regulator maps that will be used in our proofs.

Proposition 4.3 *Let w and $w_n, n \ge 1$ be as defined in Definition 4.1, and let $\gamma_n > 0$, $\gamma > 0$ be such that $\gamma_n \to \gamma$ as $n \to \infty$. Also assume the function u and the sequence of functions $\{u_n\}$ are non-negative uniformly Lipschitz continuous (with the same Lipschitz constant κ_u) and satisfies $||u_n - u||_\infty \equiv \sup_{x\in I\!R} |u_n(x) - u(x)| \to 0$, as $n \to \infty$. Then for some universal constant $\tilde{c} > 0$ (which is also independent of $u, \{u_n\}$), the following holds for all $T > 0$:*

$$
\begin{aligned}
&(a) \qquad ||\phi^{u_n,\gamma_n}(w)||_T \vee ||\psi_{2,b}^{u_n,\gamma_n}(w)||_T \ \le \ \tilde{c} ||w||_T, \ \forall\, n \ge 1. \\
&(b) \quad \text{If } \lim_{n\to 0} ||w_n - w||_T = 0, \text{ then } \ \lim_{n\to 0} ||\phi^{u_n,\gamma_n}(w_n) - \phi^{u,\gamma}(w)||_T \vee ||\psi^{u_n,\gamma_n}(w_n) - \psi^{u,\gamma}(w)||_T = 0, \\
&\qquad \lim_{n\to 0} ||\phi_b^{u_n,\gamma_n}(w_n) - \phi_b^{u,\gamma}(w)||_T \vee ||\psi_{1,b}^{u_n,\gamma_n}(w_n) - \psi_{1,b}^{u,\gamma}(w)||_T \vee ||\psi_{2,b}^{u_n,\gamma_n}(w_n) - \psi_{2,b}^{u,\gamma}(w)||_T = 0.
\end{aligned}
$$

In other words, part (b) states that for $n \to \infty$, if $w_n \to w$ uniformly on compacts (u.o.c.), then $(\phi^{u_n,\gamma_n}, \psi^{u_n,\gamma_n})(w_n) \to (\phi^{u,\gamma}, \psi^{u,\gamma})(w)$ and $(\phi_b^{u_n,\gamma_n}, \psi_{1,b}^{u_n,\gamma_n}, \psi_{2,b}^{u_n,\gamma_n})(w_n) \to (\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma})(w)$ u.o.c.

Proof. First note that from the definition of $\mathcal{M}^{u_n,\gamma_n}$ and $\mathcal{M}_b^{u_n,\gamma_n}$ in (4.2) and the fact that $u_n \ge 0$, it follows that

$$\mathcal{M}^{u_n,\gamma_n}(w)(t) \le w(t), \text{ and } \mathcal{M}_b^{u_n,\gamma_n}(w)(t) \le w(t), \ \forall\, t \ge 0.$$

Hence, by the monotonicity property of the conventional regulator maps in (4.1) (see [17]), we obtain that

$$
\begin{aligned}
0 &\le \phi^{u_n,\gamma_n}(w) \equiv \phi(\mathcal{M}^{u_n,\gamma_n}(w)) \le \phi(w), \\
0 &\le \psi_{2,b}^{u_n,\gamma_n}(w) \equiv \psi_{1,b}(\mathcal{M}_b^{u_n,\gamma_n}(w)) \le \phi(w).
\end{aligned}
$$

The proof of part (a) is now complete using Lemma 4.2 (a).

For part (b), let $\nu \equiv \mathcal{M}^{u,\gamma}(w), \nu_n \equiv \mathcal{M}^{u_n,\gamma_n}(w_n)$ satisfy the first equation of (4.2). Straightforward calculations yield that for all $t \in [0,T]$

$$\begin{aligned}
&|\nu_n(t) - \nu(t)| \\
&\quad \leq |w_n(t) - w(t)| + \int_0^t |u_n(\phi(\nu_n)(s)) - u(\phi(\nu)(s))|ds + \int_0^t |\gamma_n\phi(\nu_n)(s) - \gamma\phi(\nu)(s)|ds \\
&\quad \leq |w_n(t) - w(t)| + \int_0^t |u_n(\phi(\nu_n)(s)) - u(\phi(\nu_n)(s))|ds + \int_0^t |u(\phi(\nu_n)(s)) - u(\phi(\nu)(s))|ds \\
&\qquad\qquad + \int_0^t \gamma_n|\phi(\nu_n)(s) - \phi(\nu)(s)|ds + |\gamma_n - \gamma| \int_0^t |\phi(\nu)(s)|ds.
\end{aligned}$$

Hence, using the Lipschitz continuity of the conventional regulator map ϕ (with respect to the uniform norm on compacts, with Lipschitz constant 2) and the Lipschitz continuity of u with Lipschitz constant κ_u, we obtain

$$\begin{aligned}
||\nu_n - \nu||_t &\leq ||w_n - w||_T + T||u_n - u||_\infty + \int_0^t \kappa_u|\phi(\nu_n)(s) - \phi(\nu)(s)|ds \\
&\qquad + c_1\int_0^t |\phi(\nu_n)(s) - \phi(\nu)(s)|ds + T||\phi(\nu)||_T\,|\gamma_n - \gamma| \\
&\leq \Big[||w_n - w||_T + T||u_n - u||_\infty + T||\phi(\nu)||_T\,|\gamma_n - \gamma|\Big] \\
&\qquad + 2(\kappa_u + c_1)\int_0^t ||\nu_n - \nu||_s ds,
\end{aligned}$$

where $c_1 = sup_{n\geq 1}\{\gamma_n\}$. Thus, by Gronwall's inequality, we have for all $T > 0$

$$||\nu_n - \nu||_T \leq \Big[||w_n - w||_T + T\;||u_n - u||_\infty + T\;||\phi(\nu)||_T\;|\gamma_n - \gamma|\Big]e^{-2(\kappa_u+c_1)T}.$$

Hence if $||w_n - w||_T \to 0$ as $n \to \infty$, then $||\mathcal{M}^{u_n,\gamma_n}(w_n) - \mathcal{M}^{u,\gamma}(w)||_T \equiv \|\; \nu_n - \nu \;\|_T \to 0$ as $n \to \infty$. This, together with (4.1) and the first part of Lemma 4.2 (b), completes the proof of the first part of (b). Using an identical argument with the Lipschitz continuity of the two-sided conventional regulator map ϕ_b, one obtain the rest of (b). ■

4.2 Weak convergence analysis.

In this section, we prove the main theorem and other necessary results involving the processes introduced in Sections 2 and 3. We begin this section by giving alternative representations of such processes using the generalized regulator maps and define few other associated processes.

Using the results in Section 3 and the definition of the regulator processes in Definition 4.1, the solution of the BCP can be expressed as follows:

$$(X^*_x, L^*, U^*) = (X^*_{p,x}, L^*_p, U^*_p) = (\phi^{u^*,\gamma}_{b^*}, \psi^{u^*,\gamma}_{1,b^*}, \psi^{u^*,\gamma}_{2,b^*})(W_x), \tag{4.3}$$

where W_x is a Brownian motion starting from x as in (3.1) (see Remark 3.3 (b)). When the reference to value of the parameter p is not important, we simply identify (X^*_x, L^*, U^*) as $(X^*_{p,x}, L^*_p, U^*_p)$. We first state a general result about alternative representations of our discounted cost functions (see Lemma 3 of [32] for a similar result).

Lemma 4.4 *Let $\widetilde{J}_p(x,u,U)$ and $J_p(\underset{\sim}{\lambda},\underset{\sim}{\mu},b)$ be as defined in (3.4) and (2.14) respectively.*

(a) For any admissible policy (u,U) for the BCP defined in Definition 3.1, we have

$$\widetilde{J}_p(x,u,U) = E\left(\int_0^\infty \delta e^{-\delta t}\left\{\beta\gamma\int_0^t X_x(s)ds + \int_0^t C(u(s))ds + p\,U(t)\right\}dt\right).$$

(b) For any admissible control $(\underset{\sim}{\lambda},\underset{\sim}{\mu},b)$ for the queueing network (see Definition 2.2), we have

$$J_p(\underset{\sim}{\lambda},\underset{\sim}{\mu},b) = \liminf_{n\to\infty} E\left(\int_0^\infty \delta e^{-\delta t}\left\{\beta(n\gamma_n)\int_0^t \hat{Q}_n(s)ds + \int_0^t C(u_n(\hat{Q}_n(s)))ds + p\,\hat{U}_n(t)\right\}dt\right).$$

Proof. Note that for all $t \ge 0$,

$$e^{-\delta t} = \int_t^\infty \delta e^{-\delta s}ds = \int_{\mathbb{R}} I_{[t,\infty)}(s)\delta e^{-\delta s}ds \tag{4.4}$$

From (4.4) and the non-negativity of all the integrands below, we can interchange the order of integration using Fubini-Tonelli's theorem, and consequently we obtain

$$\begin{aligned}
&\int_0^\infty e^{-\delta t}[\{\beta\gamma X_x(t) + C(u(t))\}dt + p\,dU(t)] \\
&= \int_0^\infty\int_0^\infty I_{[t,\infty)}(s)(\delta e^{-\delta s})[\{\beta\gamma X_x(t) + C(u(t))\}dt + p\,dU(t)]ds \\
&= \int_0^\infty \delta e^{-\delta s}[\int_0^s \{\beta\gamma X_x(t) + C(u(t))\}dt + p\,U(s)]ds.
\end{aligned}$$

This proves part (a). Similar calculation yields part (b) as well. ■

Next we define the following time-change processes: For each $n \ge 1$ and $t \ge 0$, we let

$$\tau_n^A(t) \equiv \int_0^t \bar{\lambda}_n(\sqrt{n}\hat{Q}_n(s))ds,\ \tau_n^S(t) \equiv \int_0^t \bar{\mu}_n(\sqrt{n}\hat{Q}_n(s))ds,\ \tau_n^R(t) \equiv \int_0^t \gamma_n\sqrt{n}\hat{Q}_n(s)ds, \tag{4.5}$$

where $\bar{\lambda}_n(x) = \lambda_n(x)I_{\{x<\sqrt{n}b\}}, \bar{\mu}_n(x) = \mu_n(x)I_{\{x>0\}}$ are as in Section 2. Also define

$$\hat{M}_n^A(t) \equiv \hat{Y}_n^A(\tau_n^A(t)),\ \hat{M}_n^S(t) \equiv \hat{Y}_n^S(\tau_n^S(t)),\ \text{ and } \hat{M}_n^R(t) \equiv \hat{Y}_n^R(\tau_n^R(t)). \tag{4.6}$$

Then from (2.13), we have the following alternative representation of $\hat{W}_n$:

$$\hat{W}_n(t) = \hat{M}_n^A(t) - \hat{M}_n^S(t) - \hat{M}_n^R(t),\ n \ge 1, t \ge 0. \tag{4.7}$$

Using the existence, uniqueness and other properties of the generalized regulator maps in Definition 4.1, we obtain that for any admissible control $(\underset{\sim}{\lambda},\underset{\sim}{\mu},b)$, the associated processes in the queueing network have the following representation. For $n \ge 1$,

$$\begin{aligned}
(\hat{Q}_n,\hat{L}_n,\hat{U}_n) &= \left(\phi^{u_n,n\gamma_n},\psi_{1,b}^{u_n,n\gamma_n},\psi_{2,b}^{u_n,n\gamma_n}\right)(\hat{W}_n), \quad \text{if } b<\infty, \\
(\hat{Q}_n,\hat{L}_n) &= (\phi^{u_n,n\gamma_n},\psi^{u_n,n\gamma_n})(\hat{W}_n), \qquad\qquad \text{if } b=\infty.
\end{aligned} \tag{4.8}$$

We also define the following *fluid scaled* version of the processes: For $n \geq 1,\ t \geq 0$, let

$$\begin{gathered}\bar{Q}_n(t) \doteq \tfrac{1}{n} Q_n(nt) = \tfrac{1}{\sqrt{n}} \hat{Q}_n(t),\ \bar{L}_n(t) \doteq \tfrac{1}{n} L_n(nt) = \tfrac{1}{\sqrt{n}} \hat{L}_n(t), \\ \bar{U}_n(t) \doteq \tfrac{1}{n} U_n(nt) = \tfrac{1}{\sqrt{n}} \hat{U}_n(t), \ \text{ and } \bar{W}_n(t) = \tfrac{1}{\sqrt{n}} \hat{W}_n(t). \end{gathered} \tag{4.9}$$

For each $n \geq 1$ and $x \geq 0$, we let $\bar{u}_n(x) = \frac{u_n(\sqrt{n}x)}{\sqrt{n}}$. By Definition 2.2, we deduce that

$$||\bar{u}_n||_\infty = \sup_{x \geq 0} |\bar{u}_n(x)| \to 0, \ \text{ as } n \to 0. \tag{4.10}$$

Hence, from (4.9) and (2.12), we have

$$\bar{Q}_n(t) = \frac{1}{\sqrt{n}} \hat{Q}_n(t) = \bar{W}_n(t) - \int_0^t [\bar{u}_n(\bar{Q}_n(s)) + (n\gamma_n)\bar{Q}_n(s)]ds + \bar{L}_n(t) - \bar{U}_n(t), \ \ t \geq 0. \tag{4.11}$$

From the properties of the regulator maps in Definition 4.1 and (4.9), it follows that

$$\begin{aligned} (\bar{Q}_n, \bar{L}_n, \bar{U}_n) &= \left(\phi^{\bar{u}_n, n\gamma_n}, \psi_{1,b}^{\bar{u}_n, n\gamma_n}, \psi_{2,b}^{\bar{u}_n, n\gamma_n}\right)(\bar{W}_n), \ \text{ if } b < \infty, \\ (\bar{Q}_n, \bar{L}_n) &= (\phi^{\bar{u}_n, n\gamma_n}, \psi^{\bar{u}_n, n\gamma_n})(\bar{W}_n), \qquad\qquad \text{ if } b = \infty. \end{aligned} \tag{4.12}$$

The following representation also follows from (4.5) and (4.12) :

$$\bar{Q}_n(t) = \bar{W}_n(t) + [\tau_n^A(t) - \tau_n^S(t) - \tau_n^R(t)] + \bar{L}_n(t) - \bar{U}_n(t), \ \text{ for all } t \geq 0,\ n \geq 1. \tag{4.13}$$

Proposition 4.5 *Let $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ be an admissible control policy (as in Definition 2.2) for the queueing network. Let $\tau_n = (\tau_n^A, \tau_n^S, \tau_n^R), n \geq 1$ and $\tau = (\lambda e, \lambda e, 0)$, where $\tau_n^A, \tau_n^S, \tau_n^R$ are as in (4.5), $e(t) \equiv t, t \geq 0$ is the identity function and 0 denotes the function that is identically zero. Then,*

- *(a)* $\lim_{n\to\infty} \sup_{t\in[0,T]} |\tau_n(t) - \tau(t)| = 0$ *a.s. as $n \to \infty$, for all $T > 0$.*
- *(b)* $\hat{W}_n \Rightarrow W_0$ *as $n \to \infty$, where W_0 is a Brownian motion starting from zero and has infinitesimal mean and variance 0 and 2λ respectively.*
- *(c) If $b < \infty$, we let $(X_0, L, U) \doteq (\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma})(W_0)$. In the case of $b = \infty$, we define $(X_0, L, U) \doteq (\phi^{u,\gamma}(W_0), \psi^{u,\gamma}(W_0), 0)$. Then in both cases,*

  $$(\hat{Q}_n, \hat{L}_n, U_n) \Rightarrow (X_0, L, U) \ \ \text{as } n \to \infty, \tag{4.14}$$

  and (X_0, u, U) is admissible for the BCP with the initial value $x = 0$ (see Definition 3.1).
- *(d) There exists a constant $\bar{c} > 0$, such that for all $n \geq 1$ and $T > 0$*

  $$E\left[\sup_{0\leq t\leq T} |\hat{W}_n(t)|^2\right] \ \leq \ \bar{c}(T^2 + T).$$

Proof. We begin by proving part (a). As we show below, the proofs of parts (b), (c) and (d) follow from part (a). The main steps for the proof of part (a) are : we first bound the time-change processes τ_n using the functional strong law of large numbers (see (4.17) and (4.18) below). Then, this bound together with the Martingale structure of $\bar{W}_n$ implies that $\bar{W}_n \to 0$ almost surely, u.o.c. (see (4.19)). With the help of the properties of the generalized regulator maps, we complete the proof of part (a) (see (4.21), (4.22) and (4.23) below).

Fix $T > 0$. Note that from (2.1), we have

$$Q_n(t) \leq Y_n^A\left(\int_0^t \bar{\lambda}_n(Q_n(s))ds\right), \text{ for all } n \geq 1, t \geq 0,$$

where Y_n^A is as defined in (2.1). Hence, by (4.9) and (4.5), we obtain

$$0 \leq \bar{Q}_n(t) \leq \frac{Y_n^A(n\tau_n^A(t))}{n}, \text{ for all } n \geq 1, t \geq 0. \tag{4.15}$$

Note that by the functional law of large numbers for Poisson process (with intensity 1), it follows that for large n,

$$t - 1 \leq \frac{Y_n^A(nt)}{n} \leq t + 1, \text{ for all } t \in [0, cT], \tag{4.16}$$

where c is as in (2.2). Observe that $\tau_n^A(t) \leq ct \leq cT$ for all $t \in [0, T]$. Hence, by (4.15) and (4.16) we derive the following bound for large n,

$$0 \quad \leq \quad \bar{Q}_n(t) \leq c(t+1), \text{ for all } t \in [0, T].$$

Since $\sup_{n\geq 1}\{n\gamma_n\} < \infty$ (by Assumption 2.1), we get

$$\tau_n^R(t) \;=\; (n\gamma_n)\int_0^t \bar{Q}_n(s)ds \;\leq\; c_1 t(t+1)^2 \text{ for all } t \in [0, T], \tag{4.17}$$

where $c_1 > 0$ is a generic constant which is independent of n and T. Using (2.2) and (4.5), we also obtain

$$\tau_n^A(t) \leq ct, \quad \text{and } \tau_n^S(t) \leq ct, \text{ for all } t \in [0, T]. \tag{4.18}$$

By the functional strong law of large numbers for any sequence of unit-intensity independent Poisson processes $\{Y_n\}$, we have

$$\sup_{0\leq t\leq T}\left|\frac{Y_n(nt)}{n} - t\right| \to 0 \text{ a.s., as } n \to \infty.$$

The bounds in (4.17)-(4.18) together with (4.6) yields that for all $T > 0$, the following almost sure convergence results hold.

$$\begin{aligned}
0 \leq \frac{\sup_{0\leq t\leq T}|\hat{M}_n^A(t)|}{\sqrt{n}} \quad &\leq \quad \sup_{0\leq t\leq cT}\left|\frac{Y_n^A(nt)}{n} - t\right| \to 0 \text{ a.s., as } n \to \infty, \\
0 \leq \frac{\sup_{0\leq t\leq T}|\hat{M}_n^S(t)|}{\sqrt{n}} \quad &\leq \quad \sup_{0\leq t\leq cT}\left|\frac{Y_n^S(nt)}{n} - t\right| \to 0 \text{ a.s., as } n \to \infty, \\
0 \leq \frac{\sup_{0\leq t\leq T}|\hat{M}_n^R(t)|}{\sqrt{n}} \quad &\leq \quad \sup_{0\leq t\leq c_1T(T+1)}\left|\frac{Y_n^R(nt)}{n} - t\right| \to 0 \text{ a.s., as } n \to \infty.
\end{aligned}$$

Consequently, by (4.7) and (4.9) we have

$$\bar{W}_n = \frac{1}{\sqrt{n}}[\hat{M}_n^A - \hat{M}_n^S - \hat{M}_n^R] \to 0 \text{ a.s., as } n \to \infty, \tag{4.19}$$

and this convergence is uniform on compact sets. We can use (4.19), (4.10) and (4.12) together with the continuity properties of the generalized regulator maps established in Proposition 4.3 to conclude that

$$\bar{Q}_n \to 0,\ \bar{L}_n \to 0,\ \text{ and } \bar{U}_n \to 0 \text{ a.s., as } n \to \infty, \tag{4.20}$$

and this convergence is uniform on compact sets. Note that $b = \infty$ will correspond to representations of the above processes using one-sided generalized maps in (4.12) and $\bar{U}_n \equiv 0$. Hence

$$\begin{aligned}
\sup_{0\le t\le T} \left|\tau_n^A(t) - \lambda t\right| &\le \sup_{0\le t\le T} \left[\int_0^t |\bar{\lambda}_n(\sqrt{n}\hat{Q}_n(s)) - \lambda_n(\sqrt{n}\hat{Q}_n(s))|ds + \int_0^t |\lambda_n(\sqrt{n}\hat{Q}_n(s)) - \lambda|ds\right] \\
&\le \lambda_n(\sqrt{n}b)\int_0^t I_{\{\hat{Q}_n(s)=b\}}ds \ +\ T\left[\sup_{x\ge 0}|\lambda_n(x) - \lambda|\right] \\
&\le \bar{U}_n(T)\ +\ T\left[\sup_{x\ge 0}|\lambda_n(x) - \lambda|\right] \ \to\ 0 \ \text{ a.s., as } n \to \infty,
\end{aligned}$$

by (2.11), (4.20), and (2.3). This proves

$$\tau_n^A \to \lambda e \text{ a.s., as } n \to \infty, \tag{4.21}$$

uniformly on compact sets. A similar argument can be used to prove that

$$\tau_n^S \to \lambda e \text{ a.s., as } n \to \infty, \tag{4.22}$$

uniformly on compact sets. Also observe that from (4.13), one has

$$\tau_n^R(t) = \bar{W}_n(t) - \bar{Q}_n(t) + [\tau_n^A(t) - \tau_n^S(t)] + \bar{L}_n(t) - \bar{U}_n(t).$$

Hence, (4.19), (4.20), (4.21) and (4.22) together yield

$$\tau_n^R \to 0 \text{ a.s., as } n \to \infty, \tag{4.23}$$

and this convergence is uniform on compact sets. This completes the proof of part (a).

For part (b), observe that from the functional central limit theorem for Poisson processes:

$$(\hat{Y}_n^A, \hat{Y}_n^S, \hat{Y}_n^R) \Rightarrow (W^A, W^S, W^R) \text{ as } n \to \infty, \tag{4.24}$$

where W^A, W^S, W^R are three independent standard Brownian motions with mean 0 and variance t. We can use (4.6), part (a) above and the random time change theorem (see Sec. 14 of [7]) to obtain

$$(\hat{M}_n^A(\cdot), \hat{M}_n^S(\cdot), \hat{M}_n^R(\cdot)) \equiv (\hat{Y}_n^A(\tau_n^A(\cdot)), \hat{Y}_n^S(\tau_n^S(\cdot)), \hat{Y}_n^R(\tau_n^R(\cdot))) \Rightarrow (W^A(\lambda\cdot), W^S(\lambda\cdot), 0), \tag{4.25}$$

as $n \to \infty$. Here, we also use the continuity of the weak limit of $(\hat{Y}_n^A, \hat{Y}_n^S, \hat{Y}_n^S)$ and the sum (and the difference) is a continuous map on the space of continuous functions. Hence, from (4.25) and the continuous mapping theorem, we obtain

$$\hat{W}_n(\cdot) = \hat{M}_n^A(\cdot) - \hat{M}_n^S(\cdot) - \hat{M}_n^R(\cdot) \Rightarrow W^A(\lambda\cdot) - W^S(\lambda\cdot) \text{ as } n \to \infty.$$

Notice that, if we define $W_0(\cdot) \doteq W^A(\lambda\cdot) - W^S(\lambda\cdot)$, then by the independence of W^A and W^S, W_0 is a Brownian motion starting from 0 and has mean 0, variance $2\lambda t$. The proof of (b) is now complete.

To prove part (c), note that from part (b) we have

$$\hat{W}_n \Rightarrow W_0 \text{ as } n \to \infty.$$

The weak limit above is continuous and the space of all continuous functions is separable. Hence, by Skorohod representation theorem (Theorem 6.7 in [7]), one can assume that the above convergence takes place almost surely between $\{\hat{W}_n'\}, W_0'$ defined on some common probability space and $(\{W_n'\}, W_0')$ has the same law as $(\{W_n\}, W_0)$. Denoting these new elements by $(\{W_n\}, W_0)$ again (to simplify notation), we have the following convergence uniformly on compact sets.

$$W_n \to W_0 \text{ a.s., as } \ n \to \infty.$$

In the case of $b = \infty$, by (4.8) and Proposition 4.3 we obtain,

$$(\hat{Q}_n, \hat{L}_n, \hat{U}_n) = (\phi_b^{u_n, n\gamma_n}, \psi_{1,b}^{u_n, n\gamma_n}, \psi_{2,b}^{u_n, n\gamma_n})(\hat{W}_n) \to (\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma})(W_0) \doteq (X_0, L, U) \text{ a.s.,}$$

as $n \to \infty$. When $b < \infty$, with the same reasoning, we have

$$(\hat{Q}_n, \hat{L}_n) = (\phi^{u_n, n\gamma_n}, \psi^{u_n, n\gamma_n})(\hat{W}_n) \to (\phi^{u,\gamma}, \psi^{u,\gamma})(W_0) \doteq (X_0, L) \text{ a.s., as } n \to \infty. \tag{4.26}$$

Both of these convergence results hold uniformly on compact sets. Therefore, we can conclude that for each $b \in (0, \infty]$

$$(\hat{Q}_n, \hat{L}_n, U_n) \Rightarrow (X_0, L, U) \ \text{ as } n \to \infty,$$

with the convention that $\hat{U_n} = U \equiv 0$ if $b = \infty$. By the properties of the regulator maps in Definition 4.1 and the properties of W_0 in part (b), it is clear that the weak limit (X_0, L, U) satisfies the properties of the corresponding processes of the BCP (see (3.1)). Hence we conclude that the limit (X_0, u, U) is admissible for the BCP as required in Definition 3.1, and the proof of part (c) is complete.

Now we prove part (d). First observe that $\hat{Y}_n^A, \hat{Y}_n^S, \hat{Y}_n^R$ defined in (2.10) are scaled compensated Poisson processes, and hence these processes are martingales. So, by Doob's maximal inequality (Corollary 2.17 of Chapter 2 of [10]), we get for $T > 0$,

$$E\left[\sup_{0 \le t \le T} |\hat{Y}_n^A(t)|\right]^2 \le 4E\left[|\hat{Y}_n^A(T)|^2\right] = 4T.$$

Hence by (4.6) and (4.18), for all $T > 0$ the following estimate holds.

$$E\left[\sup_{0 \le t \le T} |\hat{M}_n^A(t)|\right]^2 \le 4cT. \tag{4.27}$$

Similar calculations involving $\hat{Y}_n^S, \hat{Y}_n^R$, with (4.6), (4.18) and (4.17) yield

$$E\left[\sup_{0 \le t \le T} |\hat{M}_n^S(t)|\right]^2 \le 4cT, \ \text{ and } \ E\left[\sup_{0 \le t \le T} |\hat{M}_n^R(t)|\right]^2 \le 2c_1(T+1)^2. \tag{4.28}$$

Hence, from the definition of $\hat{W}_n$ in (4.7) together with (4.27) and (4.28), we obtain that

$$E\left[\sup_{0 \le t \le T} |\hat{W}_n(t)|\right]^2 \ \le \ 4cT + 4cT + 2c_1(T+1)^2 \le C(T+1)^2, \text{ for all } T > 0,$$

where $C > 0$ is a generic constant independent of n and T. This completes the proof of part (c), and that of the proposition. ■

Theorem 4.6 *Let* $(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*)$ *be a proposed policy given in Definition 2.5. Then,*

(a) $(\hat{W}_n^*, \hat{Q}_n^*, \hat{L}_n^*, U_n^*) \Rightarrow (W_0, X_0^*, L^*, U^*)$ *as* $n \to \infty$ *where the* W_0 *is a standard Brownian motion starting from zero and* (X_0^*, L^*, U^*) *are the processes associated with the solution of the BCP with* W_0 *and the initial point* $x = 0$*, as in (3.22). Here if* $b^* = \infty$*, then* $\hat{U}_n^* = U^* \equiv 0$ *and the processes* X_0^* *and* L^* *are as described in (3.23).*

(b) $J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*) = \mathrm{V}_p(0)$ *where* $\mathrm{V}_p(x)$ *represents the value function defined in (3.5).*

Remark 4.7 *In part (a) of the above theorem, for* $(\hat{W}_n^*, \hat{Q}_n^*, \hat{L}_n^*, U_n^*)$*, we use an additional superscript* $*$ *to our notation of the queueing network processes to emphasize that these processes are obtained by using the proposed policy in Definition 2.5. Also, in part (b), for* $(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*)$*,* $J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*)$ *turned out to be the limit of the right side of (2.14) (instead of the* $\liminf$ *in (2.14)).*

Proof. Part (a) follows directly from part(c) of Proposition 4.5. We now prove part (b) using part (a). The proof is different for the different values of the cost parameter p, and is described separately in two cases.

Case I: $0 < p < p_0$**.** This case leads to an optimal finite buffer size $b^* < \infty$ as in Theorem 3.7. Note that by Assumption 2.1 and continuous mapping theorem (for the map $\eta(x)(t) = \int_0^t x(s)ds,\ t \geq 0$, $x \in \mathcal{D}([0,\infty),[0,\infty))$ and under uniform convergence on compacts), we obtain

$$\beta(n\gamma_n)\int_0^{\cdot} \hat{Q}_n^*(s)ds \ \Rightarrow\ \beta\gamma\int_0^{\cdot} X_0^*(s)ds \ \text{ a.s., as } \ n \to \infty, \tag{4.29}$$

uniformly on compact sets. Also note that since $b^* < \infty$, $0 \leq \int_0^t \hat{Q}_n^*(s)ds \leq b^* t$ for all $t \geq 0$ and (4.29) implies that for each $t \geq 0$,

$$r_1^n(t) \equiv \beta(n\gamma_n)E\left[\int_0^t \hat{Q}_n^*(s)ds\right] \to \beta\gamma E\left[\int_0^t X_0^*(s)ds\right] \ \text{as } n \to \infty. \tag{4.30}$$

Also, using Cauchy-Schwartz inequality, we derive for $c_1 = \max_{n\geq 1}\{n\gamma_n\} < \infty$, that

$$0 \leq \int_0^\infty \delta e^{-\delta t}\left[r_1^n(t)\right]^2 dt \ \leq\ [\beta^2 c_1^2 b^{*2}]\int_0^\infty \delta e^{-\delta t} t^2 dt \ <\ \infty,$$

and this bound on the right side does not involve n. Hence, we have established a sufficient condition (see (3.18) of Section 3 of [7]) for the uniform integrability to conclude (from (4.30)) that

$$\begin{aligned}
&\lim_{n\to\infty} E\int_0^\infty \delta e^{-\delta t}\left\{\beta(n\gamma_n)\int_0^t \hat{Q}_n^*(s)ds\right\}dt \\
&= \lim_{n\to\infty}\int_0^\infty \delta e^{-\delta t} r_1^n(t)dt = \int_0^\infty \delta e^{-\delta t} E\left[\beta\gamma\int_0^t X_0^*(s)ds\right]dt \\
&= E\int_0^\infty \delta e^{-\delta t}\left\{\beta\gamma\int_0^t X_0^*(s)ds\right\}dt.
\end{aligned} \tag{4.31}$$

Also combining the admissibility of the proposed control, the fact that $u_n^* = u^*$ and the properties of the cost function $C(\cdot)$ in Assumption 2.4 and part (a) above together with the continuous mapping theorem (as in (4.29)) we obtain

$$\int_0^{\cdot} C(u_n^*(\hat{Q}_n^*(s)))ds \ \Rightarrow\ \int_0^{\cdot} C(u^*(X_0^*(s)))ds \text{ as } n \to \infty. \tag{4.32}$$

For $t \geq 0$, let $r_2^n(t) \equiv E\left[\int_0^t C(u^*(X_0^*(s)))ds\right]$. Then from the fact that $0 \leq X_0^*(s) \leq b^* < \infty$ for all $s \geq 0$, it similarly follows that

$$0 \leq \int_0^\infty \delta e^{-\delta t}\left[r_2^n(t)\right]^2 dt \leq [c_2]^2 \int_0^\infty \delta e^{-\delta t} t^2 dt < \infty, \tag{4.33}$$

and this upper bound is also independent of n. Here $c_2 = \sup_{y \in [0,\bar{u}]} C(y)$, where $\bar{u} = \sup_{x \in [0,b^*]} u^*(x)$. Following the same argument as in (4.31), we obtain

$$\lim_{n\to\infty} E\int_0^\infty \delta e^{-\delta t}\left\{\int_0^t C(u_n^*(\hat{Q}_n^*(s)))ds\right\}dt = E\int_0^\infty \delta e^{-\delta t}\left\{\int_0^t C(u^*(X_0^*(s)))ds\right\}dt. \tag{4.34}$$

Note that $u_n^* \equiv u^* \geq 0$. Hence, using Assumption 2.1, (4.8), nondecreasing nature of $\hat{U}_n^*$ and the second bound in part (a) of Lemma 4.3, we have

$$0 \leq \hat{U}_n^*(t) = ||\hat{U}_n^*||_t = \left|\left|\psi_{2,b}^{u^*,n\gamma_n}\left(\hat{W}_n^*\right)\right|\right|_t \leq \tilde{c} \sup_{0\leq s\leq t} |\hat{W}_n^*(s)| \text{ for all } t \geq 0, n \geq 1,$$

for some $\tilde{c} > 0$. Hence, using part (d) of Proposition 4.5, we have for each $n \geq 1, t \geq 0$,

$$E[\hat{U}_n^*(t)]^2 \leq \tilde{c}^2 E\left[\sup_{0\leq s\leq t}|\hat{W}_n^*(s)|\right]^2 \leq [\tilde{c}^2\bar{c}](t^2+t). \tag{4.35}$$

With this upper bound and following the same approach as we used in establishing the convergence in (4.31) and (4.34), we obtain

$$r_3^n(t) \equiv pE[\hat{U}_n^*(t)] \to pE[U^*(t)], \quad \text{for all } t \geq 0 \text{ and } 0 < p < p_0. \tag{4.36}$$

Also, by (4.35) we have

$$\int_0^\infty \delta e^{-\delta t}\left[r_3^n(t)\right]^2 dt \leq p^2[\tilde{c}^2\bar{c}]\int_0^\infty \delta e^{-\delta t}(t^2+t)dt < \infty$$

and this upper bound is free of n. Thus, using a similar calculation as in (4.31) and (4.34) above, we obtain

$$\lim_{n\to\infty} E\int_0^\infty \delta e^{-\delta t}\{p\,\hat{U}_n^*(t)\}dt = E\int_0^\infty \delta e^{-\delta t}\{p\,U^*(t)\}dt. \tag{4.37}$$

Using (4.31), (4.34), (4.37), definition of the cost function in (2.14), (3.4), and (3.5), the Lemma 4.4 and the fact that W_0, the weak-limit of $\{\hat{X}_n^*\}$ is a standard Brownian motion starting at $x = 0$, we derive

$$J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*) = \widetilde{J}_p(0, u^*, U^*) = \mathrm{V}_p(0).$$

This completes the proof for the case $0 < p < p_0$.

Case II: $p \geq p_0$**.** This case leads to the optimality of the infinite buffer size $b^* = \infty$ (see Theorem 3.7). Hence, the proof of this case is somewhat straightforward, since

$$\hat{U}_n^* = U^* \equiv 0. \tag{4.38}$$

Hence the convergence of the last component of the cost function (the one dealt with in (4.37)) follows trivially. Since, $u_n^* \equiv u^* \geq 0$, using Assumption 2.1, (4.8) and the first bound in part (a) of Lemma 4.3, we obtain

$$0 \leq \sup_{0\leq s\leq t}|\hat{Q}_n^*(s)| = ||\hat{Q}_n^*||_t = \left|\left|\phi_b^{u^*,n\gamma_n}\left(\hat{W}_n^*\right)\right|\right|_t \leq \tilde{c}\sup_{0\leq s\leq t}|\hat{W}_n^*(s)|, \text{ for all } n \geq 1, t \geq 0, \tag{4.39}$$

for some $\tilde{c} > 0$. Hence, using part (d) of Proposition 4.5 and Assumption 2.2, we have for each $n \geq 1$,

$$\begin{aligned}\int_0^\infty \delta e^{-\delta t} E\left[\left\{\beta(n\gamma_n)\int_0^t \hat{Q}_n^*(s)ds\right\}^2\right]dt &\leq [\tilde{c}^2\beta^2 c_1^2]\int_0^\infty \delta e^{-\delta t} t^2 E\left[\sup_{0\leq s\leq t}|\hat{W}_n^*(s)|^2\right]dt \\ &\leq [\tilde{c}^2\beta^2 c_1^2 \bar{c}]\int_0^\infty \delta e^{-\delta t} t^2(t^2+t)dt < \infty. \qquad (4.40)\end{aligned}$$

Notice that the upper bound above does not involve n. Since (4.29) holds in this case as well, the uniform square integrability in (4.40) provides the required uniform integrability with respect to the product measure $P \times \mu$, where $d\mu/dt = \delta e^{-\delta t}$, to conclude

$$\lim_{n\to\infty} E\int_0^\infty \delta e^{-\delta t}\left\{\beta(n\gamma_n)\int_0^t \hat{Q}_n^*(s)ds\right\}dt = E\int_0^\infty \delta e^{-\delta t}\left\{\beta(\gamma)\int_0^t X_0^*(s)ds\right\}dt. \qquad (4.41)$$

Also note that the convergence results in (4.32) holds in this case as well. Recall that, by our definition of optimal drift $u^* = u_p^*$ in (3.24) of Theorem 3.7, we have

$$u^*(x) = \Psi(\mathrm{V}_p'(x)) \text{ and } 0 \leq \mathrm{V}_p'(x) < p_0,$$

and Ψ is a nondecreasing function with $\Psi(p_0) = \theta_{p_0} < \infty$ (see (3.11) and the discussion above that). Hence,

$$0 \leq u_n^*(x) \equiv u^*(x) \leq \theta_{p_0}, \text{ for all } x \geq 0.$$

Let $c_2 \doteq \sup_{y\in[0,\theta_{p_0}]} C(y)$. Using the above bound, we can obtain the same bound as in (4.33). Hence using (4.34), we conclude that (arguing as in (4.34)) that

$$\lim_{n\to\infty} E\int_0^\infty \delta e^{-\delta t}\left\{\int_0^t C(u_n^*(\hat{Q}_n^*(s)))ds\right\}dt = E\int_0^\infty \delta e^{-\delta t}\left\{\int_0^t C(u^*(X_0^*(s)))ds\right\}dt. \qquad (4.42)$$

Using (4.38), (4.41), (4.42), the definition of the cost functional in (2.14), (3.4), and (3.5), the Lemma 4.4 and the fact that W_0, the weak-limit of $\{\hat{X}_n^*\}$, is a Brownian motion starting at $x = 0$, we derive that

$$J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*) = \widetilde{J}_p(0, u^*, U^*) = \mathrm{V}_p(0).$$

This completes the proof for $p \geq p_0$. ■

Proof of Theorem 2.6:

Theorem 4.6 proves that

$$J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b) = \mathrm{V}_p(0).$$

Hence, it is enough to prove that if $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ is any admissible policy satisfying Definition 2.2, then

$$J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \geq \mathrm{V}_p(0). \qquad (4.43)$$

Note that (4.43) holds trivially if $J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) = \infty$. Hence, we will assume

$$J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) < \infty \qquad (4.44)$$

and intend to verify (4.43). Using Assumption 2.4, (2.5), part (c) of Proposition 4.5 and the Skorohod representation theorem, it follows that

$$\left(\int_0^{\cdot} \hat{Q}_n(s)ds, \int_0^{\cdot} C(u_n(\hat{Q}_n(s)))ds\right) \to \left(\int_0^{\cdot} X_0(s)ds, \int_0^{\cdot} C(u(X_0(s)))ds\right) \text{ a.s., as } n \to \infty, \quad (4.45)$$

uniformly on compact sets (see (4.29) and (4.32) for a similar argument). Using part (b) of Lemma 4.4, (4.14) and applying Fatou's lemma twice , we derive

$$\begin{aligned} J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) &= \liminf_{n\to\infty} E\int_0^{\infty} \delta e^{-\delta t}\left\{\beta(n\gamma_n)\int_0^t \hat{Q}_n(s)ds + \int_0^t C(u(\hat{Q}_n(s)))ds + p\, U_n(t)\right\} ds \\ &\geq E\int_0^{\infty} \delta e^{-\delta t}\left[\lim_{n\to\infty}\left\{\beta(n\gamma_n)\int_0^t \hat{Q}_n(s)ds + \int_0^t C(u_n(\hat{Q}_n(s)))ds + p\, U_n(t)\right\}\right] dt \\ &= E\int_0^{\infty} \delta e^{-\delta t}\left\{\beta\gamma\int_0^t X_0(s)ds + \int_0^t C(u(X_0(s)))ds + p\, U(t)\right\} dt \qquad (4.46) \end{aligned}$$

where X_0 and U are as defined in part (c) of Proposition 4.5. As shown in Proposition 4.5 (c), (X_0, u, U) is an admissible control of the BCP (with W_0). Hence, using part (a) of Lemma 4.4 (3.5), we have

$$E\int_0^{\infty} \delta e^{-\delta t}\left\{\beta\gamma\int_0^t X_0(s)ds + \int_0^t C(u(X_0(s)))ds + pU(t)\right\} dt \geq \widetilde{J}_p(0, u, U) \geq \mathrm{V}_p(0). \quad (4.47)$$

Thus we get from (4.46) - (4.47) that

$$J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \geq \mathrm{V}_p(0).$$

and the proof of the theorem is complete. ■

Remark 4.8 (Computing b^* numerically) *For the given cost structure in (2.14), when* $0 < p < p_0$*, we can compute the finite buffer size* b^* *numerically by an algorithm very similar to the one described in Section 5 of [13] (see also [19] for a different approach).*

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