Random linear recursions with dependent coefficients *

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Abstract

We consider the stochastic difference equation $R_n = Q_n + M_n R_{n-1}$, $n \in \mathbb{Z}$, for Markov-dependent coefficients $(Q_n, M_n) \in \mathbb{R}^2$. Under natural conditions on the coefficients, and regardless of the initial value R_0, the series $(R_n)_{n\geq 0}$ converges in distribution to its long-term equilibrium R, which is the unique stationary solution of this equation.

We study the asymptotic behavior of the distribution tails of R and show that both $P(R > t)$ and $P(R < -t)$ are regularly varying at infinity. This extends previous work [12, 13] done for independent identically distributed coefficients to a more general setup, which is more realistic in some real-world applications and is desirable in certain theoretical models.

In addition, using a Markovian representation of certain chains with complete connections [2, 20], we apply the results of this paper and of [26] to coefficients induced by such processes.

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1 Introduction and statement of results

Let $(Q_n, M_n)_{n\in\mathbb{Z}}$ be a stationary and ergodic sequence of pairs of real-valued random variables such that $E(\log|M_0|) < 0$ and $E\left(\log^+|Q_0|\right) < \infty$ (where x^+ means $\max\{x,0\}$ for $x \in \mathbb{R}$) and consider the recurrence relation

$$R_n = Q_n + M_n R_{n-1}, \qquad n \in \mathbb{N},\ R_n \in \mathbb{R}. \tag{1}$$

This equation describes the evolution of systems in discrete time and has many real world and theoretical applications. For example, it has provided models for investment portfolios, environmental pollution dynamics, and was applied to random walks in random environments and stochastic learning algorithms. See for instance [7, 25, 27] and references therein for more examples.

For any initial random value R_0, the limit law of R_n is the same as that of

$$R = Q_0 + \sum_{n=1}^{\infty} Q_{-n} \prod_{i=0}^{n-1} M_{-i}, \tag{2}$$

and it is the unique initial distribution under which $(R_n)_{n\geq 0}$ is stationary [5]. The distribution tails of R were studied in [10, 11, 12, 13, 15, 19] under the common assumption that the pairs $(Q_n, M_n)_{n\in\mathbb{Z}}$ form an i.i.d. sequence, and in [6, 26] (see also Section 2.3 in [22]) in a Markovian setup. The tails have been shown to be regularly varying in [10, 12, 13, 19]. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called regularly varying if $f(t) = t^\alpha L(t)$ for some $\alpha \in \mathbb{R}$ and a slowly varying function $L(t)$, that is $L(\lambda t) \sim L(t)$ as $t \to \infty$ for all $\lambda > 0$. Here and henceforth $f(t) \sim g(t)$ as $t \to \infty$ means $\lim_{t\to\infty} f(t)/g(t) = 1$. The work of Kesten [19] (an alternative proof of the main result was provided by Goldie in [10]) assumes M_n to be dominant in determining the tail behavior of R, while Grey [12] and Grincevičius [13] work under the assumption that Q_n is dominant. In [11] it is shown that if $P(|M_n| \leq 1) = 1$, then the tails of R decay at least exponentially fast. In addition, [11] investigated the relationship between the tails of R and the behavior of M_n near 1 (see also a recent work [15]).

The main purpose of this paper is an extension of the results of [12, 13] in the case that the sequence $(Q_n, M_n)_{n\in\mathbb{Z}}$ is induced by a stationary countable Markov chain $(X_n)_{n\in\mathbb{Z}}$ in the following sense:

Definition 1.1. *The coefficients $(Q_n, M_n)_{n\in\mathbb{Z}}$ are said to be induced by a sequence of random variables $(X_n)_{n\in\mathbb{Z}}$, each valued in a countable set $\mathcal{D}$, if there exists a sequence of independent random variables $(Q_{n,i}, M_{n,i})_{n\in\mathbb{Z}, i\in\mathcal{D}} \in \mathbb{R}^2$ such that for a fixed $i \in \mathcal{D}$, $(Q_{n,i}, M_{n,i})_{n\in\mathbb{Z}}$ are i.i.d and*

$$Q_n = \sum_{j\in\mathcal{D}} Q_{n,j} \mathbf{I}_{\{X_n=j\}} = Q_{n,X_n} \qquad \text{and} \qquad M_n = \sum_{j\in\mathcal{D}} M_{n,j} \mathbf{I}_{\{X_n=j\}} = M_{n,X_n}. \tag{3}$$

Thus the randomness of the coefficients (Q_n, M_n) is due to two factors: 1) an underlying auxiliary process $(X_n)_{n\in\mathbb{Z}}$, which can be thought as representative of the "state of nature," and, given the value of X_n, 2) "intrinsic" random characteristics of the system, an additive factor Q_{n,X_n} and a multiplicative factor M_{n,X_n}. Allowing countable state space $\mathcal{D}$ in our setup in particular enables us to apply later on our results for Markov-dependent coefficients to a more general class of processes, namely the ones induced by chains of infinite order (see a detailed discussion below in this Introduction). An extension to a non-i.i.d. setup seems to be desirable in many, especially financial applications of (1). See for instance [24], in particular the concluding discussion therein.

We notice that an extension of the main result of [10, 19] to a Markovian setup has been obtained in [6, 26]. The mechanisms leading to heavy (regularly varying) tails of R are quite different in [10, 19] versus [12, 13]. In the first case, there exists a (uniquely defined) critical exponent α such that $E(|M_n|^\alpha) = 1$ and $E(|Q_n|^\alpha) < \infty$ which determines some of important asymptotic properties of the random walk $S_n = \sum_{i=0}^{n-1} \log |M_{-i}|$ and, correspondingly, of $\prod_{i=0}^{n-1} |M_{-i}| = e^{S_n}$. In this case, R is a sum of rapidly decaying, in general light-tailed terms, and is heavy tailed essentially because of one atypical fluctuation of $\prod_{i=0}^{n} |M_{-i}|$. In the second case, the critical exponent is not available, and therefore more explicit assumptions about distribution tails of Q_n are made. Then Q_n dominates and creates cumulative effects, namely R turns out to be heavy tailed as a sum of heavy-tailed terms $Q_{-n} \prod_{i=0}^{n-1} M_{-i}$.

The following assumption for $(Q_{n,i}, M_{n,i})$ will be used in Section 2.

Assumption 1.2. *Let $\mathcal{D}$ be a countable set. There exists a constant $\alpha > 0$ such that:*

(A1) There are a function $L(t)$ slowly varying at infinity and two sequences of nonnegative numbers $(q_+(i))_{i\in\mathcal{D}}$ and $(q_-(i))_{i\in\mathcal{D}}$ such that, for all $n \in \mathbb{Z}$ and $i \in \mathcal{D}$, we have

$$\lim_{t\to\infty} \frac{P(Q_{n,i} > t)}{t^{-\alpha}L(t)} = q_+(i) \quad \text{and} \quad \lim_{t\to\infty} \frac{P(Q_{n,i} < -t)}{t^{-\alpha}L(t)} = q_-(i),$$

and in both cases the convergence is uniform on $i \in \mathcal{D}$. Furthermore, we assume that

$$\sum_{j\in\mathcal{D}} q_+(j) > 0 \qquad \text{and} \qquad \sup_{j\in\mathcal{D}} \max\{q_+(j), q_-(j)\} < \infty. \tag{4}$$

(A2) There exist constant $\beta \in (0, \alpha)$ and $\mu > 0$ such that for all $i \in \mathcal{D}$, $E(|M_{0,i}|^\beta) < \mu < \infty$.

(A3) For $i \in \mathcal{D}$, let

$$m_+(i) = E\big(|M_{0,i}|^\alpha \mathbf{I}_{\{M_{0,i}>0\}}\big), \quad m_-(i) = E\big(|M_{0,i}|^\alpha \mathbf{I}_{\{M_{0,i}<0\}}\big),$$

and $m(i) = m_+(i) + m_-(i)$. Then there exists a constant $\lambda \in (0, 1)$ such that $m(i) < \lambda$ for all $i \in \mathcal{D}$.

(A4) If $\mathcal{D}$ is an infinite countable set, then $\lim_{\varepsilon\to 0+} P(M_{1,i} \le \varepsilon) = P(M_{1,i} \le 0)$ uniformly on $i \in \mathcal{D}$ (the assumption of course trivially holds if $\mathcal{D}$ is a finite set).

The assumptions for the tail behavior of Q_n in Assumption 1.2 essentially determine the tail behavior of R. The remaining assumptions are technical and are for the purposes of facilitating uniformity of asymptotic behavior of terms involved in our asymptotic analysis.

Let $B_b = \{(x_n)_{n\in\mathcal{D}} : x_n \in \mathbb{R} \text{ and } \sup_{n\in\mathcal{D}} |x_n| < +\infty\}$ be the space of bounded elements of $\mathbb{R}^{\mathcal{D}}$. Assumption 1.2 implies that the following vectors belong to the space B_b :

$$\begin{aligned} q_+ := \big(q_+(i)\big)_{i\in\mathcal{D}}, \quad q_- := \big(q_-(i)\big)_{i\in\mathcal{D}}, \qquad m_+ := \big(m_+(i)\big)_{i\in\mathcal{D}}, \quad m_- := \big(m_-(i)\big)_{i\in\mathcal{D}}, \\ q := q_+ + q_-, \qquad m := m_+ + m_-. \end{aligned} \tag{5}$$

Moreover, according to (4), $q_+ \neq \mathbf{0}$. Here and henceforth we use notation $\mathbf{0}$ and $\mathbf{1}$ for vectors (sequences) in $\mathbb{R}^{\mathcal{D}}$ whose components are respectively, all 0 or 1. We write $\mathbf{x} > \mathbf{y}$ for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathcal{D}}$ if the inequality holds for each component of the vectors (similar rules are applied for $\mathbf{x} < \mathbf{y}$, $\mathbf{x} \leq \mathbf{y}$, and $\mathbf{x} \geq \mathbf{y}$).

If $(X_n)_{n\in\mathbb{Z}}$ is a stationary irreducible Markov chain, let $\big(H(i,j)\big)_{i,j\in\mathcal{D}}$ be transition kernel of the **backward** chain $(X_{-n})_{n\in\mathbb{Z}}$ (that is $H(i,j) = P(X_n = j|X_{n+1} = i)$), and define matrices G_+, G_-, and G as

$$G_\pm(i,j) = m_\pm(i)H(i,j) \quad \text{and} \quad G(i,j) = m(i)H(i,j) \qquad \forall\, i,j \in \mathcal{D}. \tag{6}$$

Notice that *(A3)* of Assumption 1.2 implies that the spectral radius of the matrix G (as the operator acting in B_b) is strictly less than one. Throughout the paper we donote by $\pi = \big(\pi(i)\big)$ the (unique) stationary distribution of the underlying Markov chain. That is $\pi(j) = \sum_i \pi(i)H(i,j)$ for all states j.

The following theorem extends the main result of [13] and [12] to Markov-dependent coefficients (Q_n, M_n). The result shows that under Assumption 1.2, assuming in addition that $P(M_0 > 0) = 1$, the upper tail of the distribution of R is regularly varying and has the same asymptotic structure as the upper tail of Q_0. The proof of the theorem is included in Section 2.

Theorem 1.3. *Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary irreducible Markov chain on a countable state space $\mathcal{D}$. Let Assumptions 1.2 hold and suppose that $P(M_{0,i} > 0) = 1$ for all $i \in \mathcal{D}$. Then as $t \to \infty$, for all $i \in \mathcal{D}$,*

$$P(R > t|X_0 = i) \;\sim K(i)t^{-\alpha}L(t),$$

where the vector $K = \big(K(i)\big)_{i\in\mathcal{D}} \in B_b$ is defined by $K = (I - G)^{-1}q_+$, with G given by (6).

Similarly to the case where the coefficients (Q_n, M_n) of the recursion (1) are i.i.d. random variables (see [12, 13]), Theorem 1.3 yields a similar result without the restriction $P(M_0 > 0) = 1$. The proof of the following theorem is included in Section 3.

Theorem 1.4. *Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary irreducible Markov chain on a countable state space $\mathcal{D}$, $(Q_n, M_n)_{n\in\mathbb{Z}}$ be defined by (3), and let Assumption 1.2 hold. Then as $t \to \infty$, for all $i \in \mathcal{D}$,*

$$P(R > t|X_0 = i) \;\sim K_+(i)t^{-\alpha}L(t) \quad \text{and} \quad P(R < -t|X_0 = i) \;\sim K_-(i)t^{-\alpha}L(t),$$

where the vectors $K_+ = (K_+(i))_{i\in\mathcal{D}}$ and $K_- = (K_-(i))_{i\in\mathcal{D}}$ are defined by

$$\begin{aligned} K_+ &= \frac{1}{2}\Big((I-G)^{-1}q + (I - G_+ + G_-)^{-1}(q_+ - q_-)\Big) \\ &= (I - G_+ + G_-)^{-1}\big(q_+ + G_-(I-G)^{-1}q\big), \end{aligned}$$

and

$$\begin{aligned} K_- &= \frac{1}{2}\Big((I-G)^{-1}q - (I - G_+ + G_-)^{-1}(q_+ - q_-)\Big) \\ &= (I - G_+ + G_-)^{-1}\big(q_- + G_-(I-G)^{-1}q\big). \end{aligned}$$

For the sake of flexibility, we desire to formulate the tail behavior of R for when the current state of the system is dependent on possibly the entire past, with a decreasing weight on past states. It is well-known (see e.g. [2, 9, 20]) that a broad class of random processes with "fading memory" can be realized as functions of a countable positive recurrent Markov chain. In Section 4, by using the Markov representation obtained by Lalley [20], we prove a counterpart of Theorem 1.4 for the so called chains of infinite order. The result of Lalley shows that for certain class of chains of infinite order $(X_n)_{n\in\mathbb{Z}}$ there exist a stationary irreducible Markov chain $(Y_n)_{n\in\mathbb{Z}}$ defined on a countable state space $\mathcal{S}$ and a function $\zeta : \mathcal{S} \to \mathcal{D}$ such that

$$(X_n)_{n\in\mathbb{Z}} \stackrel{D}{=} \big(\zeta(Y_n)\big)_{n\in\mathbb{Z}}, \tag{7}$$

where $\stackrel{D}{=}$ means equivalence of distributions. More precisely, following [20] we define:

Definition 1.5. *[20] A C-chain is a stationary process $(X_n)_{n\in\mathbb{Z}}$ taking values in a finite set (alphabet) $\mathcal{D}$ such that*

(i) For any $i_1, i_2, \ldots, i_n \in \mathcal{D}$,

$$P(X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n) > 0. \tag{8}$$

(ii) For any $i_0 \in \mathcal{D}$ and any sequence $(i_n)_{n\geq 1} \in \mathcal{D}^{\mathbb{N}}$, the following limit exists:

$$\lim_{n\to\infty} P(X_0 = i_0 | X_{-k} = i_k,\ 1 \leq k \leq n) = P(X_0 = i_0 | X_{-k} = i_k,\ k \geq 1), \tag{9}$$

where the right-hand side in (9) is a regular version of the conditional probabilities with respect to $\sigma(X_n : n > 0)$.

(iii) For $n \geq 0$ let

$$\gamma_n := \sup\left\{ \left| \frac{P(X_0 = i_0 | X_{-k} = i_k,\ k \geq 1)}{P(X_0 = j_0 | X_{-k} = j_k,\ k \geq 1)} - 1 \right| :\ i_k, j_k \in \mathcal{D} \text{ and } i_k = j_k,\ k = 1, \ldots, n \right\}.$$

Then, the numbers γ_n are all finite and $\limsup_n \log \gamma_n / n < 0$.

C-chains are a sub-class of two related general classes of random processes. On one hand, C-chains are a particular case of chains of infinite order (also known as chains with complete connections). For general accounts on chains with complete connections we refer to [8, 16, 17], see also [9] and references therein for some recent developments. On the other hand, the stationary distributions of C-chains (a particular case of g-measures introduced by Keane [18]) are Gibbs states in the sense of Bowen (also known as Dobrushin-Lanford-Ruelle (DLR) states), see e.g. [4, 21] and the discussion in [20]. We have:

Theorem A (Lalley [20]). *Let $(X_n)_{n\in\mathbb{Z}}$ be a C-chain on an alphabet $\mathcal{D}$. Let $\mathcal{S} = \bigcup_{n\geq 1} \mathcal{D}^n$ and $\zeta : \mathcal{S} \to \mathcal{D}$ be the projection into the last coordinate, i.e. $\zeta\big((s_1, \ldots, s_n)\big) = s_n$. Then there exist a stationary irreducible Markov chain $(Y_n)_{n\in\mathbb{Z}}$ defined on the state space $\mathcal{S}$ such that*

(i) The Markov representation (7) holds.

(ii) $P\big(Y_{n+1} = (x_1, x_2, \ldots, x_t)|Y_n = (y_1, y_2, \ldots, y_s)\big) = 0$ *unless either* $t = 1$ *or* $t = s + 1$ *and* $x_i = y_i$ *for all* $i \le r$.

(iii) $P\big(Y_{n+1} = (y)|Y_{n+1} \in \mathcal{D},\, Y_n = (y_1, y_2, \ldots, y_s)\big) = P\big(Y_0 = (y)|Y_0 \in \mathcal{D}\big)$ *for any elements* $y, y_1, \ldots, y_s \in \mathcal{D}$.

(iv) There exist constants $r \in \mathbb{N}$ *and* $\delta > 0$ *such that* $P\big(Y_{n+1} \in \mathcal{D}|Y_n = (y_1, y_2, \ldots, y_{mr})\big) = \delta$, *for all* $m \in \mathbb{N}$ *and all* $(y_1, y_2, \ldots, y_{mr}) \in \mathcal{D}^{mr}$.

(v) Without loss of generality we can assume that r *in (iv) is* ***even*** *(see the beginning of the proof of Theorem 1 on page 1266 in [20]).*

We assume that, if needed, the underlying probability space is enlarged, to include the (stationary under law P) sequence $(Y_n)_{n\in\mathbb{Z}}$ and that

$$X_n = \zeta(Y_n), \qquad n \in \mathbb{Z}. \tag{10}$$

In fact, the sequences $(X_n)_{n\in\mathbb{Z}}$ and $(Y_n)_{n\in\mathbb{Z}}$ are explicitly coupled in the construction of [20]. Notice that the process Y_n (Lalley [20] calls it a "list process") is a sequence of words, where the next word is usually obtained by the concatenation of the previous one with a random symbol from the alphabet. However, at certain random times the process "forgets" its history and cuts the list by setting $Y_{n+1} = (X_{n+1})$.

For C-chains, using the Markovian representation (7), we deduce from Theorem 1.4 the following result. Let

$$\mathcal{X} = (X_n)_{n\ge 0}. \tag{11}$$

Under our assumptions the sequence $\mathcal{X}$ serves as a "random environment" for the underlying system whose evolution in time is described by (1). In what follows, we will often use the fact that due to (7), a statement which is valid for P-almost every sequence $\mathcal{Y} = (Y_n)_{n\ge 0}$ is true also for P-almost every sequence $\mathcal{X}$.

We have (here and henceforth we use standard notations $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \ge 0\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \ge 0\}$):

Theorem 1.6. *Let* $(X_n)_{n\in\mathbb{Z}}$ *be a C-chain,* $(Q_n, M_n)_{n\in\mathbb{Z}}$ *be defined by* (3), *and let Assumption 1.2 hold. Then, for some* ***bounded*** *functions* $K_+ : \mathcal{D}^{\mathbb{Z}_+} \to \mathbb{R}_+$ *and* $K_- : \mathcal{D}^{\mathbb{Z}_+} \to \mathbb{R}_+$, *as* $t \to \infty$,

$$P(R > t|\mathcal{X}) \;\sim K_+(\mathcal{X})t^{-\alpha}L(t) \quad \textit{and} \quad P(R < -t|\mathcal{X}) \;\sim K_-(\mathcal{X})t^{-\alpha}L(t), \qquad P-a.s.$$

Moreover, $P\big(K_+(\mathcal{X}) \ne 0\big) > 0$.

The result is a trivial consequence of Theorem 1.4 combined with the existence of a Markovian representation (10). Indeed, using the fact that the state space $\mathcal{D}$ in the definition of C-chains is finite, it is not hard to check that the conditions of Theorem 1.4 are satisfied by the positive recurrent countable Markov chain Y_n, and the coefficients $(\widetilde{Q}_n, \widetilde{M}_n)_{n\in\mathbb{Z}}$ induced by this chain and defined by the following formulas:

$$\widetilde{Q}_{n,i} = Q_{n,\zeta(i)} \qquad \text{and} \qquad \widetilde{M}_{n,i} = M_{n,\zeta(i)}. \tag{12}$$

In particular, the expressions for K_+ and K_- in Theorem 1.6 are similar to those in Theorem 1.4, with G being defined in terms of the transition kernel of the backward Markov chain $(Y_{-n})_{n\in\mathbb{Z}}$.

To complete the picture we also derive an extension of the main result of [19] and [10] to C-chains. The next theorem is obtained in Section 4 rather directly from its i.i.d. prototype (cf. [19, Theorem 5], see also [10, Theorem 4.1]) by applying Doeblin's "cyclic trick" to the Markov chain Y_n defined in (7) and associating (2) with an equivalent linear recursion model having i.i.d. coefficients. Similar results for a Markovian setup can be found in [6, 26], see also Section 2.3 in [22]. In principle, we could derive the theorem for C-chains by a direct reduction to the main results of [26]. However, we prefer to give in Section 4 a full alternative proof based on the ideas of [22] and appealing only to some of technical results in [26] because of the following two reasons: 1) The proof for coefficients induced by countable Markov chain given here is much simpler than the one needed for a general state space in [26]; 2) The Markov chain Y_n defined in (7) as well as the definition (3) of the coefficients (Q_n, M_n) make the setup discussed in this paper somewhat special and therefore allow to simplify significantly certain conditions of the theorems in [26]. Therefore, some extra work would be anyway needed in order to derive Theorem 1.7 below directly from the results in [26]. For the sake of both transparency and completeness we prefer to provide a relatively short and almost self-contained proof of the following theorem in this paper.

Recall the random variable $\mathcal{X}$ from (11). We have:

Theorem 1.7. *Let $(X_n)_{n\in\mathbb{Z}}$ be a C-chain, $(Q_n, M_n)_{n\in\mathbb{Z}}$ be defined by (3), and assume that*

(i) $P(|Q_0| < q_0) = 1$ for some constant $q_0 > 0$.

(ii) $P(m_0^{-1} < |M_0| < m_0) = 1$ for some constant $m_0 > 1$.

(iii) $\limsup_{n\to\infty} \frac{1}{n} \log E\left(\prod_{i=0}^{n-1} |M_i|^{\beta_1}\right) \geq 0$ and $\limsup_{n\to\infty} \frac{1}{n} \log E\left(\prod_{i=0}^{n-1} |M_i|^{\beta_2}\right) < 0$ for some constants $\beta_1 > 0$ and $\beta_2 > 0$.

(iv) $P(\log|M_0| \in \delta\cdot\mathbb{Z} | M_0 \neq 0) < 1$ for all $\delta > 0$, where $\delta\cdot\mathbb{Z}$ is the scaled by δ integer lattice $\{0, \pm\delta, \pm 2\delta, \ldots\}$.

Then, with parameter $\alpha > 0$ defined below in (14), the following properties hold:

(a) The following limits exist with probability one:

$$\lim_{t\to\infty} t^\alpha P(R > t|\mathcal{X}) = K_1(\mathcal{X}) \quad \text{and} \quad \lim_{t\to\infty} t^\alpha P(R < -t|\mathcal{X}) = K_{-1}(\mathcal{X})$$

*for some **bounded** functions $K_{-1} : \mathcal{D}^{\mathbb{Z}_+} \to \mathbb{R}_+$ and $K_1 : \mathcal{D}^{\mathbb{Z}_+} \to \mathbb{R}_+$, as $t \to \infty$. Moreover, the convergence is uniform on $\mathcal{X}$.*

(b) If $P(M_0 < 0) > 0$ then $P\big(K_1(\mathcal{X}) = K_{-1}(\mathcal{X})\big) = 1$.

(c) If $P(Q_0 > 0, M_0 > 0) = 1$ then $P\big(K_1(\mathcal{X}) > 0\big) = 1$.

(d) For $\eta \in \{-1, 1\}$, if $P\big(K_\eta(\mathcal{X}) > 0\big) > 0$ then $P\big(K_\eta(\mathcal{X}) > 0\big) = 1$.

(e) $P\big(K_1(\mathcal{X}) = K_{-1}(\mathcal{X}) = 0\big) = 1$ *if and only if there exists a function* $\Gamma : \mathcal{D} \to \mathbb{R}$ *such that*

$$P\big(Q_0 + \Gamma(X_0)M_0 = \Gamma(X_1)\big) = 1. \tag{13}$$

Moreover, if the random pairs $(Q_{0,i}, M_{0,i})$ *are non-degenerate for every* $i \in \mathcal{D}$*, that is if* $P\big(Q_{0,i} = q, M_{0,i} = m\big) < 1$ *for any pair of constants* $(q, m) \in \mathbb{R}^2$ *and any* $i \in \mathcal{D}$*, then* $P\big(K_1(\mathcal{X}) = K_{-1}(\mathcal{X}) = 0\big) = 1$ *if and only if there exists a constant* $c \in \mathbb{R}$ *such that* $P\big(Q_0 + cM_0 = c\big) = 1$.

Note that the inequality with β_2 in condition (iii) of the theorem implies by Jensen's inequality that $E(\log|M_0|) < 0$. Thus, the series in (2) converges absolutely, P-a.s. Proposition 4.1 below states that both lim sup's in (iii) are in fact limits, and the parameter $\alpha > 0$ is (uniquely) determined by

$$\Lambda(\alpha) = 0\,, \quad \text{where } \Lambda(\beta) = \lim_{n\to\infty} \frac{1}{n} \log E\left(\textstyle\prod_{i=0}^{n-1} |M_i|^\beta\right). \tag{14}$$

In order to show that $P\big(K_1(\mathcal{X}) + K_{-1}(\mathcal{X}) > 0\big) \neq 0$, we need an extra non-degeneracy assumption (namely, that (13) is false for any Γ) which guarantees that the random variable R is not a deterministic function of the initial state X_0. This condition is a natural generalization of the criterion that appears in the case where the random variables $(Q_n, M_n)_{n\in\mathbb{Z}}$ are i.i.d. (cf. [19] and [10]).

The rest of the paper is organized as follows. Section 2 contains the proof of Theorem 1.3, Section 3 includes the proof of Theorem 1.6, while the proof of Theorem 1.7 is included in Section 4.

2 Proof of Theorem 1.3

This section is devoted to the proof of Theoem 1.3. The proof is by a series of lemmas. The key to the result is Proposition 2.2 which extends the corresponding statement for i.i.d. coefficients (Q_n, M_n) considered in [12] and [13]. Throughout this section, let Assumptions 1.2 hold and suppose in addition that $P(M_{0,i} > 0) = 1$ for all $i \in \mathcal{D}$.

First, we need the following trivial extension of the weak convergence of R_n to R.

Lemma 2.1. *For any* $i \in \mathcal{D}$ *and* $t \in \mathbb{R}$, $\lim_{n\to\infty} P(R_n \le t | X_n = i) = P(R \le t | X_0 = i)$ *for any point of continuity* t *of* $P(R \le t | X_0 = i)$.

The above result follows using exactly the same arguments as in [5], where the weak convergence of R_n to R is shown. We therefore omit details and refer the reader to [5]. Notice that the explicit expression $R = Q_0 + Q_{-1}M_0 + Q_{-2}M_0M_1 + \ldots$ along with (3) imply that the sequence $(R_n, X_n)_{n\in\mathbb{Z}}$ is stationary and ergodic in the case $R_0 = R$.

The next proposition, which extends Lemma 2 of [12], is the cornerstone element of our proof of Theorem 1.3. See the discussion in the paragraph right after Lemma 2.4 below.

Proposition 2.2. *Let* Y *be a random variable such that:*

(i) $Y \in \sigma\big(X_n, (Q_{n,i}, M_{n,i}) : n \le 0, i \in \mathcal{D}\big)$.

(ii) There exist a positive number $C > 0$ and non-negative constants $(c_i)_{i\in\mathcal{D}}$, such that

(a) $\lim_{t\to\infty} \frac{P(Y>t|X_0=i)}{t^{-\alpha}L(t)} = c_i$ *for every* $i \in \mathcal{D}$*, and, furthermore, the convergence is uniform on* i*.*

(b) We have:

$$P(Y_j > t) < C \cdot t^{-\alpha} L(t) \quad \text{for any } i \in \mathcal{D} \text{ and all } t > 0. \tag{15}$$

(c) $\sum_{i\in\mathcal{D}} c_i > 0$*, that is not all the coefficients* c_i *are zero.*

(iii) There exist a positive number $\widetilde{C} > 0$ and non-negative constants $(\tilde{c}_i)_{i\in\mathcal{D}}$, such that

(a) $\lim_{t\to\infty} \frac{P(Y<-t|X_0=i)}{t^{-\alpha}L(t)} = \tilde{c}_i$ *for every* $i \in \mathcal{D}$*, and, furthermore, the convergence is uniform on* i*.*

(b) $P(Y < -t|X_0 = i) \le \widetilde{C} \cdot t^{-\alpha} L(t)$ *for all* $i \in \mathcal{D}$ *and* $t > 0$*.*

Then for all $i \in \mathcal{D}$*,*

$$\lim_{t\to\infty} \frac{P(Q_1 + M_1 Y > t|X_1 = i)}{t^{-\alpha}L(t)} = q_+(i) + m_+(i) \sum_{j\in\mathcal{D}} H(i,j) c_j. \tag{16}$$

Moreover, the convergence is uniform on $i \in \mathcal{D}$*.*

Proof. Let $(Y_i)_{i\in\mathcal{D}}$ be a sequence of random variables independent of the sequences $(X_n)_{n\in\mathbb{Z}}$ and $(Q_{1,i}, M_{1,i})_{i\in\mathcal{D}}$, and such that $P(Y_i > t) = P(Y > t|X_0 = i)$ for all $t > 0$. We shall use the fact that the random variables Y and (Q_1, M_1) are independent given X_1 and X_0, and moreover $P\big((Q_1, M_1, Y) \in \cdot|X_1 = i, X_0 = j\big) = P\big((Q_{1,i}, M_{1,i}, Y_j) \in \cdot\big)$ for all $i, j \in \mathcal{D}$. In particular, for any $i \in \mathcal{D}$,

$$\begin{aligned} P(Q_1 + M_1 Y > t, X_1 = i) &= \sum_{j\in\mathcal{D}} P(Q_1 + M_1 Y > t, X_1 = i, X_0 = j) \\ &= \sum_{j\in\mathcal{D}} P(Q_1 + M_1 Y > t|X_1 = i, X_0 = j)\pi_i H(i,j) \\ &= \sum_{j\in\mathcal{D}} P(Q_{1,i} + M_{1,i} Y_j > t)\pi_i H(i,j). \end{aligned} \tag{17}$$

By Lemma 2 of [12], which is the i.i.d. prototype of our proposition, we have for each $j \in \mathcal{D}$,

$$P(Q_{1,i} + M_{1,i} Y_j > t) \sim t^{-\alpha} L(t)\big(q_+(i) + c_j m_+(i)\big) \text{ as } t \to \infty. \tag{18}$$

To complete the proof of the proposition it is enough to show that the above convergence is uniform both on i and j. Indeed, the uniform convergence in (18) along with (17) imply that

$$P(Q_1 + M_1 Y > t|X_1 = i) \sim t^{-\alpha} L(t) \sum_{j\in\mathcal{D}} \big(q_+(i) + c_j m_+(i)\big) H(i,j) \text{ uniformly on } i \in \mathcal{D},$$

as desired.

To prove that the convergence in (18) is uniform on $i, j \in \mathcal{D}$, we will use a decomposition of $P(Q_{1,i} + M_{1,i}Y_j > t)$ into four individually tractable terms, similar to the one exploited in Lemma 2 of [12]. Namely, we fix $\varepsilon > 0$ and write

$$A_{i,j}(t) := P(Q_{1,i} + M_{1,i}Y_j > t) = A_{i,j}^{(1)}(t) - A_{i,j}^{(2)}(t) + A_{i,j}^{(3)}(t) + A_{i,j}^{(4)}(t),$$

where

$$\begin{aligned}
A_{i,j}^{(1)}(t) &= P\big(Q_{1,i} > t(1+\varepsilon)\big) \\
A_{i,j}^{(2)}(t) &= P\big(Q_{1,i} > (1+\varepsilon)t,\, Q_{1,i} + M_{1,i}Y_j \le t\big) \\
A_{i,j}^{(3)}(t) &= P\big((1-\varepsilon)t < Q_{1,i} \le (1+\varepsilon)t,\, Q_{1,i} + M_{1,i}Y_j > t\big) \\
A_{i,j}^{(4)}(t) &= P\big(Q_{1,i} \le (1-\varepsilon)t,\, Q_{1,i} + M_{1,i}Y_j > t\big)
\end{aligned}$$

It follows from *(A1)* of Assumption 1.2 (more precisely from the asymptotic behavior of the upper tail of Q_0 and the upper bound on $q_+(i)$ included in (4)) that $\frac{A_{i,j}^{(1)}}{t^{-\alpha}L(t)}$ converges uniformly to $q_+(i)(1+\varepsilon)^{-\alpha}$.

For $A_{i,j}^{(2)}$ we use the bound

$$\begin{aligned}
0 &\le A_{i,j}^{(2)}(t) = P\big(Q_{1,i} > (1+\varepsilon)t,\, Q_{1,i} + M_{1,i}Y_j \le t;\, Y_j < -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big) \\
&+ P\big(Q_{1,i} > (1+\varepsilon)t,\, Q_{1,i} + M_{1,i}Y_j \le t;\, Y_j \ge -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big) \\
&\le P\big(M_{1,i}Y_j \le -\varepsilon t,\, Y_j \ge -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big) + P\big(Q_{1,i} > (1+\varepsilon)t,\, Y_j < -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big) \\
&\le P\big(M_{1,i} \ge t^{\frac{\alpha+\beta}{2\beta}}\big) + P\big(Q_{1,i} > (1+\varepsilon)t\big)P\big(Y_j < -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big),
\end{aligned}$$

where in the last step we used the independence of $Q_{1,i}$ and Y_j.

By Chebyshev's inequality $P\big(M_{1,i} \ge t^{\frac{\alpha+\beta}{2\beta}}\big) \le E\big(M_{1,i}^{\beta}\big) \cdot t^{-\frac{\alpha+\beta}{2}}$, which yields a uniform on i bound on $P\big(M_{1,i} \ge t^{\frac{\alpha+\beta}{2\beta}}\big)$ due to *(A1)* of Assumption 1.2. Furthermore, by the condition (iii)-b of the proposition, for all $i \in \mathcal{D}$ and t large enough we have

$$P\big(Y_j < -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big) \le \widetilde{C}\big(\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big)^{-\alpha} L\big(\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big),$$

which shows that $P\big(Y_j < -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big)$ tends to zero uniformly on j when $t \to \infty$.

To bound $A_{i,j}^{(3)}$ we write

$$\begin{aligned}
A_{i,j}^{(3)}(t) &\le P\big((1-\varepsilon)t < Q_{1,i} \le (1+\varepsilon)t\big) \le P\big(Q_{1,i} > (1-\varepsilon)t\big) - P\big(Q_{1,i} > (1+\varepsilon)t\big) \\
&\sim q_+(i)\big[(1-\varepsilon)^{-\alpha} - (1+\varepsilon)^{-\alpha}\big],
\end{aligned}$$

which in view of (4) yields a uniform bound for $A_{i,j}^{(3)}(t)$ which tends to zero when $\varepsilon \to 0$.

It remains to show the asymptotic for $A_{i,j}^{(4)}$. Fix constants $m > 0$ and $n > 0$, and let

$A^{(4)}_{i,j}(t) = A^{(4,1)}_{i,j}(t) + A^{(4,2)}_{i,j}(t) + A^{(4,3)}_{i,j}(t)$ where

$$\begin{aligned}
A^{(4,1)}_{i,j}(t) &= E\Big(\frac{P\big(Y_j > M_{1,i}^{-1}(t-Q_{1,i})\big)}{t^{-\alpha}L(t)}\mathbf{I}_{\{Q_{1,i}\le(1-\varepsilon)t\}}\mathbf{I}_{\{M_{1,i}>m\}}\Big)\\
A^{(4,2)}_{i,j}(t) &= E\Big(\frac{P\big(Y_j > M_{1,i}^{-1}(t-Q_{1,i})\big)}{t^{-\alpha}L(t)}\mathbf{I}_{\{Q_{1,i}\le(1-\varepsilon)t\}}\mathbf{I}_{\{M_{1,i}\le m\}}\mathbf{I}_{\{|Q_{1,i}|>n\}}\Big)\\
A^{(4,3)}_{i,j}(t) &= E\Big(\frac{P\big(Y_j > M_{1,i}^{-1}(t-Q_{1,i})\big)}{t^{-\alpha}L(t)}\mathbf{I}_{\{Q_{1,i}\le(1-\varepsilon)t\}}\mathbf{I}_{\{M_{1,i}\le m\}}\mathbf{I}_{\{|Q_{1,i}|\le n\}}\Big).
\end{aligned}$$

We will now show that the first two terms tend to zero uniformly as n and m go to infinity. First note that

$$\begin{aligned}
A^{(4,1)}_{i,j}(t) &\le E\Big(\frac{P\big(Y_j > M_{1,i}^{-1}\varepsilon t\big)}{t^{-\alpha}L(t)}\mathbf{I}_{\{M_{1,i}>m\}}\Big)\\
&= E\Big(\frac{P\big(Y_j > M_{1,i}^{-1}\varepsilon t\big)}{(M_{1,i}^{-1}\varepsilon t)^{-\alpha}L(M_{1,i}^{-1}\varepsilon t)}\frac{(M_{1,i}^{-1}\varepsilon t)^{-\alpha}L(M_{1,i}^{-1}\varepsilon t)}{t^{-\alpha}L(t)}\mathbf{I}_{\{M_{1,i}>m\}}\Big). \qquad (19)
\end{aligned}$$

Therefore (15) along with (19) imply that

$$A^{(4,1)}_{i,j}(t) \le E\Big(\frac{CM^{\alpha}_{1,i}}{\varepsilon^{\alpha}}\cdot\frac{L(M_{1,i}^{-1}\varepsilon t)}{L(t)}\mathbf{I}_{\{M_i>m\}}\Big)$$

By Lemma 1 of [12], for a given $\delta > 0$ there exists $K = K(\delta) > 0$ such that

$$\frac{L(\lambda t)}{L(t)} \le \max\{\lambda^{\alpha}, K\lambda^{-\delta}\}, \qquad \forall\ \lambda > 0,\ t > 0,$$

and therefore, applying with result to $\lambda = M_{1,i}^{-1}\varepsilon$, we obtain

$$\begin{aligned}
A^{(4,1)}_{i,j}(t) &\le E\left(\frac{CM^{\alpha}_{1,i}}{\varepsilon^{\alpha}}\max\{(M_{1,i}^{-1}\varepsilon)^{\alpha}, K(M_{1,i}^{-1}\varepsilon)^{-\delta}\}\mathbf{I}_{\{M_{1,i}>m\}}\right)\\
&\le E\Big(\frac{CM^{\alpha}_{1,i}}{\varepsilon^{\alpha}}(M_{1,i}^{-\alpha}\varepsilon^{\alpha} + KM^{\delta}_{1,i}\varepsilon^{-\delta})\mathbf{I}_{\{M_{1,i}>m\}}\Big).
\end{aligned}$$

Setting $\delta = \frac{\beta-\alpha}{2} > 0$ and applying Hölder inequality with $p = \frac{2\beta}{\alpha+\beta}$ and $q = \frac{2\beta}{\beta-\alpha}$ to the right-most expression above gives (we assume, actually without loss of generality that $\varepsilon < 1$)

$$\begin{aligned}
A^{(4,1)}_{i,j}(t) &\le \frac{C}{\varepsilon^{\frac{\alpha+\beta}{2}}}E\Big(\big(1+KM_{1,i}^{\frac{\beta+\alpha}{2}}\big)\mathbf{I}_{\{M_{1,i}>m\}}\Big)\\
&\le C\varepsilon^{-\frac{\alpha+\beta}{2}}E\Big(\big(1+KM_{1,i}^{\frac{\beta+\alpha}{2}}\big)^{\frac{2\beta}{\alpha+\beta}}\Big)^{\frac{\alpha+\beta}{2\beta}}P(M_{1,i}>m)^{\frac{\beta-\alpha}{2\beta}}\\
&\le C\varepsilon^{-\frac{\alpha+\beta}{2}}E\Big(\big(1+K^{\frac{2\beta}{\beta+\alpha}}M^{\beta}_{1,i}\big)\cdot 2^{\frac{\beta-\alpha}{\alpha+\beta}}\Big)^{\frac{\alpha+\beta}{2\beta}}P(M_{1,i}>m)^{\frac{\beta-\alpha}{2\beta}},
\end{aligned}$$

where in the last step we used inequality $(x+y)^p \le 2^{p-1}(x^p+y^p)$ which is valid for any $x, y \ge 0$ and $p > 1$. The latter is in fact Jensen's inequality $\big[E(X)\big]^p \le E\big(X^p\big)$ applied to a random variable X taking values x and y with equal probabilities.

Since $P(M_{1,i} > m) \le m^{-\beta} E\big(M_{1,i}^{\beta}\big)$, it follows from *(A2)* of Assumption 1.2 that $A_{i,j}^{(4,1)}(t)$ is uniformly bounded for all $t > 0$ by a function of m and $i \in \mathcal{D}$ which tends to zero uniformly on $i \in \mathcal{D}$ when m goes to infinity.

The term $A_{i,j}^{(4,2)}(t)$ is treated similarly, and we therefore omit the details.

To show the asymptotic of $A_{i,j}^{(4,3)}(t)$ we let $f_t^{(j)}(a,b) := P\big(Y_j > a^{-1}(t-b)\big)$ for any $j \in \mathcal{D}$, $a \in \mathbb{R}, b > 0$, and $t > 0$, and write, similarly to (19),

$$\begin{aligned}
&E\Big(\frac{P\big(Y_j > M_{1,i}^{-1}(t-Q_{1,i})\big)}{t^{-\alpha}L(t)}\mathbf{I}_{\{Q_{1,i}\le(1-\varepsilon)t\}}\mathbf{I}_{\{M_{1,i}\le m\}}\mathbf{I}_{\{|Q_{1,i}|\le n\}}\Big)\\
&\quad = E\Big(\frac{f_t^{(j)}(Q_{1,i},M_{1,i})}{\big(M_{1,i}^{-1}(t-Q_{1,i})\big)^{-\alpha}L\big(M_{1,i}^{-1}(t-Q_{1,i})\big)}\times\\
&\qquad\qquad \frac{\big(M_{1,i}^{-1}(t-Q_{1,i})\big)^{-\alpha}L\big(M_{1,i}^{-1}(t-Q_{1,i})\big)}{t^{-\alpha}L(t)}\mathbf{I}_{\{Q_{1,i}\le(1-\varepsilon)t\}}\mathbf{I}_{\{M_{1,i}\le m\}}\mathbf{I}_{\{|Q_{1,i}|\le n\}}\Big).
\end{aligned}$$

Under the conditions $Q_{1,i} \le (1-\varepsilon)t$ and $M_{1,i} \le m$, we have $M_{1,i}^{-1}(t-Q_{1,i}) \ge m^{-1}\varepsilon t$. Therefore, in virtue of *(A1)* of Assumption 1.2 and condition (ii) of the proposition, the following holds true with probability one, uniformly on $i,j \in \mathcal{D}$,

$$\frac{f_t^{(j)}(Q_{1,i},M_{1,i})}{\big(M_{1,i}^{-1}(t-Q_{1,i})\big)^{-\alpha}L\big(M_{1,i}^{-1}(t-Q_{1,i})\big)}\mathbf{I}_{\{Q_{1,i}\le(1-\varepsilon)t\}}\mathbf{I}_{\{M_{1,i}\le m\}} \to_{t\to\infty} c_j\mathbf{I}_{\{M_{1,i}\le m\}}. \tag{20}$$

Further, with probability one, uniformly on $i \in \mathcal{D}$,

$$\frac{\big(M_{1,i}^{-1}(t-Q_{1,i})\big)^{-\alpha}}{t^{-\alpha}}\mathbf{I}_{\{M_{1,i}\le m\}}\mathbf{I}_{\{|Q_{1,i}|\le n\}} \to_{t\to\infty} M_{1,i}^{\alpha}\mathbf{I}_{\{M_{1,i}\le m\}}\mathbf{I}_{\{|Q_{1,i}|\le n\}}. \tag{21}$$

Next, by Theorem 1.2.1 in [3], $L(\lambda t)/L(t) \to_{t\to\infty} 1$ uniformly on compact λ-subsets of $(0,\infty)$, and hence

$$\frac{L\big(M_{1,i}^{-1}(t-Q_{1,i})\big)}{L(t)}\mathbf{I}_{\{m^{-1}<M_{1,i}\le m\}}\mathbf{I}_{\{|Q_{1,i}|\le n\}} \to_{t\to\infty} \mathbf{I}_{\{m^{-1}<M_{1,i}\le m\}}\mathbf{I}_{\{|Q_{1,i}|\le n\}}, \tag{22}$$

uniformly on $i \in \mathcal{D}$, with probability one.

Finally, assuming without loss of generality that $m > 1$, we obtain using Potter's bounds on $L(\lambda t)/L(t)$ (see Theorem 1.5.6 in [3]) that for any $\delta > 0$ there exists $t_0 = t_0(\delta)$ such that

$$\frac{L\big(M_{1,i}^{-1}(t-Q_{1,i})\big)}{L(t)}\mathbf{I}_{\{M_{1,i}\le m^{-1}\}}\mathbf{I}_{\{|Q_{1,i}|\le n\}} \le \frac{mt}{t-n}\cdot\mathbf{I}_{\{M_{1,i}\le m^{-1}\}}, \tag{23}$$

for all $t > t_0$.

Estimates (20)-(23) combined together with *(A4)* of Assumption 1.2 and the bounded convergence theorem show that $\lim_{t\to\infty} A_{i,j}^{(4)}(t) = c_jE\big(M_{1,i}^{\alpha}\big)$ uniformly on $i,j \in \mathcal{D}$. This completes proof of the proposition. □

Remark 2.3. *We note that condition (ii)-b of the proposition would essentially follow from (ii)-a and the assumption that* $\sup_{i\in\mathcal{D}} c_i < B$ *for some* $B > 0$. *Similarly, (iii)-b is essentially contained in (iii)-a combined together with an extra assumption that* $\sup_{i\in\mathcal{D}} \tilde{c}_i < \widetilde{B}$ *for some* $\widetilde{B} > 0$. *Indeed, if* $\frac{P(Y_j>t)}{t^{-\alpha}L(t)} \to_{t\to\infty} c_j < B$ *uniformly on* $j \in \mathcal{D}$, *then there is a constant* $t_0 > 0$ *such that for all* $t > t_0$

$$\frac{P(Y_j > t)}{t^{-\alpha}L(t)} < B + 1.$$

Also for $t \in (0, t_0]$,

$$\frac{P(Y_j > t)}{t^{-\alpha}L(t)} \le \frac{t^\alpha}{L(t)} \le \frac{t_0^\alpha}{L(t)}.$$

Without loss of generality we could assume that $L(t) < B$ *for all* $t \in (0, t_0]$. *Indeed, since we are only interested in asymptotic properties of* $L(t)$ *throughout, we can modify* $L(t)$ *on the interval* $(0, t_0]$ *in such a way that it would be bounded on this interval and still satisfy the conditions of Theorem 1.3. Letting* $C = \max\{1 + B, Bt_0^\alpha\}$ *we obtain* (15).

Proposition 2.2, namely asymptotic relation (16), allows us to find the asymptotic behavior of the upper tail of the distribution R_1 whenever R_0 satisfies the conditions of the proposition. To enable us to iterate the process and find the asymptotic behavior of $P(R_n > t|X_n = i)$ we need the following lemma.

Lemma 2.4. *Let* Y *be a random variable that satisfies the conditions of Proposition 2.2, and let* $\widetilde{Y} = Q_1 + M_1 Y$. *Then* $\widetilde{Y} \in \sigma\big(X_n, (Q_{n,i}, M_{n,i}) : n \le 1, i \in \mathcal{D}\big)$ *and* $\widetilde{Y}$ *also satisfies conditions (ii)-(iii) of the proposition.*

Proof.
(i) Follows directly from the definition of $\widetilde{Y}$ and condition (i) of Proposition 2.2.

(ii)-a The claim is included in conclusion (16) of Proposition 2.2.

(ii)-b Let $\tilde{\mathbf{c}} \in B_b$ be a vector defined by $\tilde{\mathbf{c}}(i) = q_+(i) + m_+(i)\sum_{j\in\mathcal{D}} H(i,j)c_j$. That is, $\tilde{\mathbf{c}}(i)$ represents the value on the right-hand side of (16). Then, in vector form,

$$\tilde{\mathbf{c}} = q_+ + G\mathbf{c},$$

where matrix G is introduced in (6) whereas vector $\mathbf{c}$ is defined by $\mathbf{c} = (c_i)_{i\in\mathcal{D}}$. Note that $\tilde{\mathbf{c}} \in B_b$ in view of assumptions (ii)-a and (ii)-b of Proposition 2.2 (see also Remark 2.3 above). Therefore, the claim follows from the definition of G, the uniform bound on $q_+(i)$ included in (4), and the uniform bound on $m_+(i)$ included in *(A3)* of Assumption 1.2.

(ii)-c Follows for instance from the inequality $\tilde{\mathbf{c}} = q_+ + G\mathbf{c} \ge q_+$ and the analogous claim about q_+ contained in (4).

(iii) The claim follows from the proof of **(ii)**, applied to the triple $(-Y, -Q_1, M_1)$ instead of (Y, Q_1, M_1). □

Proposition 2.2 together with Lemma 2.4 provide some intuition why Theorem 1.3 holds true in the Markov-dependent setup of this paper. Indeed, if Y is the random variable as in the conditions of the proposition, and we set $R_0 = Y$, then the iteration of (16) shows that $P(R_n > t|X_n = i) \sim c_{n,i}L(t)t^{-\alpha}$, where $\mathbf{c}_0 = \mathbf{c} = (c_i)_{i\in\mathcal{D}}$ and $\mathbf{c}_n = q_+ + G\mathbf{c}_{n-1}$ for all $n \geq 1$. That is, the upper tails of R_n are regularly varying. Moreover, it is not hard to check that the vectors $\mathbf{c}_n$ converge in B_b, as n goes to infinity, to vector $K = (I - G)^{-1}q_+$.

The above discussion can be summarized in the identity

$$\lim_{n\to\infty}\lim_{t\to\infty}\frac{P(R_n > t|X_n = i)}{t^{-\alpha}L(t)} = K(i). \tag{24}$$

One is prompted to show that interchanging the limits above does not change the result, namely

$$\lim_{t\to\infty}\lim_{n\to\infty}\frac{P(R_n > t|X_n = i)}{t^{-\alpha}L(t)} = K(i). \tag{25}$$

Assume for simplicity that $P(R > t|X_0 = i)$ is a continuous function of t for each $i \in \mathcal{D}$. Then by Lemma 2.1, $\lim_{n\to\infty} P(R_n > t|X_n = i) = P(R > t|X_0 = i)$. Therefore, (25) is equivalent to

$$\lim_{t\to\infty}\frac{P(R > t|X_0 = i)}{t^{-\alpha}L(t)} = K(i),$$

which is the content of Theorem 1.3.

Following the proof structure of [12], the actual proof will progress in the following way. We will bound the tails of R from above and below by the tails of sequences R_n whose tail behavior is consistent with (24). In one case we will construct a sequence R_n such that $P(R_n > t|X_n = i) \geq P(R > t|X_0 = i)$ for any $n \in \mathbb{N}$, $i \in \mathcal{D}$, and $t > 0$. In the second case we will have $P(R_n > t|X_n = i) \leq P(R > t|X_0 = i)$. In virtue of Proposition 2.2, the first construction implies $\limsup_{t\to\infty} P(R > t|X_0 = i) \leq K(i)$ while the second yields $\liminf_{t\to\infty} P(R > t|X_0 = i) \geq K(i)$.

We start to build the first construction with the following lemma. The random variable Z defined in the statement of the lemma will play the role of R_0 in this case.

Lemma 2.5. *There exists a random variable Z such that*

(i) Z is independent of $\sigma(X_n : n \in \mathbb{Z}) \vee \sigma\big((Q_{n,i}, M_{n,i}) : n \in \mathbb{Z}, i \in \mathcal{D}\big)$

(ii) There is a positive constant $c^* > 0$ such that $P(Z > t) \sim c^*L(t)t^{-\alpha}$ as $t \to \infty$

(iii) $P(Z > 0) = 1$.

(iv) $P(Q_1 + M_1Z > t|X_1 = i) \leq P(Z > t|X_0 = i)$ for all $t > 0$ and $i \in \mathcal{D}$.

Proof. One can use Z defined in the proof of Lemma 3 of [12] provided that c^* used in that construction satisfies condition $c^* > \max_{i\in\mathcal{D}}\big(1 - E(M_{0,i}^\alpha)\big)^{-1}$. The latter maximum exists due to the uniform on $i \in \mathcal{D}$ bound put on $E(M_{0,i}^\alpha)$ by *(A3)* of Assumption 1.2. □

Remark 2.6. *It is not hard to check that Z, as defined in the conditions of the lemma, satisfies all the conditions of Proposition 2.2.*

In the following result we observe that under our assumptions $P(R > 0)$ is necessarily strictly positive.

Lemma 2.7. $P(R > 0) > 0$.

Proof. First, note that R is not degenerate at zero, that is $P(R = 0) \neq 1$. Indeed, otherwise the identity $P(Q_1 + M_1 R > t) = P(R > t)$ would imply that $P(Q_1 = 0) = 1$ which contradicts the first part of (4).

Next, observe that $P(R \le 0) = 1$ would imply $P(R \le 0|X_0 = i) = 1$ and hence $P(Q_{1,i} + M_{1,i}R \le 0) = 1$ for all $i \in \mathcal{D}$. Therefore, from independence of R from $Q_{1,i}$ and $M_{1,i}$ we would get that $P(Q_{1,i} + M_{1,i}t_0 \le 0) = 1$ for some $t_0 < 0$ (in fact, for any $t_0 < 0$ such that $P(t < R) > 0$).

The last assertion contradicts our basic assumption that the upper tail of $M_{1,i}$ is of smaller order that of $Q_{1,i}$. Indeed, $1 = P(Q_{1,i} + M_{1,i}t_0 \le 0) = P(Q_{1,i}|t_0|^{-1} \le M_{1,i})$ would imply that $P(M_{1,i} > t) \ge P(Q_{1,i}|t_0|^{-1} > t) \sim (t|t_0|)^{-\alpha} L(t|t_0|)$, which is impossible in virtue of *(A2)* of Assumption 1.2. The proof of the lemma is completed. $\square$

We are now in position to complete the proof of Theorem 1.3.

Lemma 2.8. *For all* $i \in \mathcal{D}$, *we have* $\limsup_{t\to\infty} \dfrac{P(R > t|X_0 = i)}{t^{-\alpha}L(t)} \le (I - G)^{-1}q_+(i)$.

Proof. Let $R_0 = Z$, where Z is defined in the statement of Lemma 2.5. For the sequence R_n defined by the recurrence equation (1) let $R_{n,i}$ be a random variable independent of the sigma-algebra generated by $X_n, Q_{n,i}, M_{n,i}$ with $n \in \mathbb{Z}$ and $i \in \mathcal{D}$, and distributed the same as R_n under the conditional law $P(\cdot\, |X_n = i)$.

By Lemma 2.5, $P(R_1 > t|X_1 = i) \le P(R_0 > t|X_0 = i)$ for all $t > 0$ and $i \in \mathcal{D}$. This yields:

$$\begin{aligned}
P(R_2 > t, X_2 = i) &= P(Q_2 + M_2 R_1 > t, X_2 = i) \\
&= \sum_{j\in\mathcal{D}} P(Q_2 + M_2 R_1 > t|X_2 = i, X_1 = j)\pi_i H(i,j) \\
&= \sum_{j\in\mathcal{D}} P(Q_{2,i} + M_{2,i}R_{1,j} > t)\pi_i H(i,j) \le \sum_{j\in\mathcal{D}} P(Q_{1,i} + M_{1,i}R_{0,j} > t)\pi_i H(i,j) \\
&= P(Q_1 + M_1 R_0 > t, X_1 = i) = P(R_1 > t, X_1 = i).
\end{aligned}$$

Therefore $P(R_2 > t|X_2 = i) \le P(R_1 > t|X_1 = i)$ for all $i \in \mathcal{D}$ and $t > 0$. Iterating the argument, we obtain that for all $n \in \mathbb{N}$, $i \in \mathcal{D}$, and $t > 0$,

$$P(R_n > t|X_n = i) \le P(R_{n-1} > t|X_{n-1} = i). \tag{26}$$

Moreover, Proposition 2.2 and Lemma 2.4 imply that

$$P(R_n > t|X_n = i) \sim t^{-\alpha}L(t)\big(q_+ + Gq_+ + \ldots + G^{n-1}q_+ + c^* G^n \mathbf{1}\big)(i), \tag{27}$$

where constant c^* is introduced in the statement of Lemma 2.5. It follows from Lemma 2.1 and (26) that $P(R_n > t|X_n = i) \ge P(R > t|X_0 = i)$ for $n \ge 0$. Hence by (27), for all $i \in \mathcal{D}$,

$$\limsup_{t\to\infty} \frac{P(R > t|X_0 = i)}{t^{-\alpha}L(t)} \le (I - G)^{-1}q_+(i), \tag{28}$$

as required. $\square$

Lemma 2.9. *For all $i \in \mathcal{D}$, we have* $\liminf_{t\to\infty} \dfrac{P(R > t|X_0 = i)}{t^{-\alpha}L(t)} \geq (I-G)^{-1}q_+(i)$.

Proof. Let $\mathcal{R} = Q_{-1} + M_{-1}Q_{-2} + M_{-1}M_{-2}Q_{-3} + \ldots$ be the "shifted by one unit to the left" random variable R. By Lemma 2.7, for any $i \in \mathcal{D}$ and $t \geq 0$,

$$
\begin{aligned}
P(R > t) = P(Q_0 + M_0\mathcal{R} > t) &\geq \sum_{i\in\mathcal{D}} P(Q_0 > t, \mathcal{R} > 0|X_0 = i)\pi_i \\
&= \sum_{i\in\mathcal{D}} P(Q_{0,i} > t)P(\mathcal{R} > 0|X_0 = i)\pi_i \neq 0.
\end{aligned}
$$

Let R_0 be a random variable independent of $(X_n, Q_{i,n}, M_{i,n})_{n\geq 1, i\in\mathcal{D}}$ such that

$$
P(R_0 > t) = \begin{cases} P(\mathcal{R} > 0, Q_0 > t) & \text{if } t \geq 0, \\ 1 & \text{if } t < 0. \end{cases}
$$

Notice, that since

$$
P(R_0 > t|X_0 = i) = P(\mathcal{R} > 0, Q_0 > t|X_0 = i) = P(\mathcal{R} > 0|X_0 = i)P(Q_{0,i} > t), \tag{29}
$$

R_0 satisfies the conditions of Proposition 2.2 (see in particular Remark 2.3).

It follows from the definition of R_0 that, for any $i \in \mathcal{D}$ and $t > 0$,

$$
\begin{aligned}
P(R_0 > t|X_0 = i) &= P(\mathcal{R} > 0, Q_0 > t|X_0 = i) \leq P(Q_0 + M_0\mathcal{R} > t|X_0 = i) \\
&= P(R > t|X_0 = i).
\end{aligned}
$$

We will now use induction on n to show that

$$
P(R_n > t|X_n = i) \leq P(R_0 > t|X_0 = i) \qquad \forall\, n \in \mathbb{N},
$$

and hence

$$
P(R_n > t|X_n = i) \leq P(R > t|X_0 = i) \qquad \forall\, n \in \mathbb{N}. \tag{30}
$$

Specifically, given that $P(R_{n-1} > t|X_{n-1} = i) \leq P(R > t|X_0 = i)$ for some $n > 1$ and $i \in \mathcal{D}$, we obtain

$$
\begin{aligned}
P(R_n > t|X_n = i) &= \sum_{j\in\mathcal{D}} P(R_n > t, X_n = i, X_{n-1} = j)\frac{1}{\pi_i} \\
&= \sum_{j\in\mathcal{D}} P(R_n > t|X_n = i, X_{n-1} = j)H(i,j) \\
&= \sum_{j\in\mathcal{D}} P(Q_n + M_nR_{n-1} > t|X_n = i, X_{n-1} = j)H(i,j) \\
&= \sum_{j\in\mathcal{D}} P(Q_{n,i} + M_{n,i}R_{n-1} > t|X_0 = j)H(i,j) \\
&\leq \sum_{j\in\mathcal{D}} P(Q_{1,i} + M_{1,i}R > t|X_0 = j)H(i,j) \\
&= \sum_{j\in\mathcal{D}} P(Q_1 + M_1R > t|X_1 = i, X_0 = j)H(i,j) \\
&= P(Q_1 + M_1R > t|X_1 = i) = P(R > t|X_0 = i),
\end{aligned}
$$

where we used (i) the fact that $(Q_{n,i}, M_{n,i})$ and $(Q_{1,i}, M_{1,i})$ are i.i.d. for $i \in \mathcal{D}$ and (ii) the stationary property of (X_n).

Moreover, (29) implies that as t goes to infinity, uniformly on $i \in \mathcal{D}$,

$$\begin{aligned}\frac{P(R_0 > t|X_0 = i)}{t^{-\alpha}L(t)} &= \frac{P(\mathcal{R} > 0, Q_0 > t|X_0 = i)}{t^{-\alpha}L(t)} \\ &= \frac{P(Q_{0,i} > t)P(\mathcal{R} > 0|X_0 = i)}{t^{-\alpha}L(t)} \sim q_+(i)P(\mathcal{R} > 0|X_0 = i).\end{aligned}$$

Let $e_i = q_+(i)P(\mathcal{R} > 0|X_0 = i)$ and define vector $\mathbf{e} \in B_b$ by setting $\mathbf{e} = (e_i)_{i\in\mathcal{D}}$. Then Proposition 2.2 and Lemma 2.4 imply that

$$P(R_n > t|X_n = i) \sim t^{-\alpha}L(t)\big(q_+ + Gq_+ + \ldots + G^{n-1}q_+ + G^n\mathbf{e}\big)(i).$$

This completes the proof of Lemma 2.9 in view of (30). □

Lemmas 2.8 and 2.9 combined together yield Theorem 1.3.

3 Proof of Theorem 1.4

We will deduce Theorem 1.4 from Theorem 1.3 by using the following result, which extends Lemma 4 of [12]. Recall the definition of G_+ and G_- from (6).

Lemma 3.1. *Let Y be a random variable such that:*

(i) $Y \in \sigma\big(X_n, (Q_{n,i}, M_{n,i}) : n \leq 0, i \in \mathcal{D}\big)$.

(ii) $c_i := \limsup_{t\to\infty} \frac{P(Y>t|X_0=i)}{t^{-\alpha}L(t)} < \infty$ *and* $d_i := \liminf_{t\to\infty} \frac{P(Y>t|X_0=i)}{t^{-\alpha}L(t)} > -\infty$ *for every* $i \in \mathcal{D}$.

(iii) $\tilde{c}_i := \limsup_{t\to\infty} \frac{P(Y<-t|X_0=i)}{t^{-\alpha}L(t)} < \infty$ *and* $\tilde{d}_i := \liminf_{t\to\infty} \frac{P(Y<-t|X_0=i)}{t^{-\alpha}L(t)} > -\infty$ *for every* $i \in \mathcal{D}$.

Then for all $i \in \mathcal{D}$,

$$\begin{aligned}
\limsup_{t\to\infty} \frac{P(Q_1 + M_1Y > t|X_1 = i)}{t^{-\alpha}L(t)} &\leq q_+(i) + \sum_{j\in\mathcal{D}} G_+(i,j)c_j + \sum_{j\in\mathcal{D}} G_-(i,j)\tilde{c}_j, \\
\limsup_{t\to\infty} \frac{P(Q_1 + M_1Y < -t|X_1 = i)}{t^{-\alpha}L(t)} &\leq q_-(i) + \sum_{j\in\mathcal{D}} G_+(i,j)\tilde{c}_j + \sum_{j\in\mathcal{D}} G_-(i,j)c_j, \\
\liminf_{t\to\infty} \frac{P(Q_1 + M_1Y > t|X_1 = i)}{t^{-\alpha}L(t)} &\geq q_+(i) + \sum_{j\in\mathcal{D}} G_+(i,j)d_j + \sum_{j\in\mathcal{D}} G_-(i,j)\tilde{d}_j, \\
\liminf_{t\to\infty} \frac{P(Q_1 + M_1Y < -t|X_1 = i)}{t^{-\alpha}L(t)} &\geq q_-(i) + \sum_{j\in\mathcal{D}} G_+(i,j)\tilde{d}_j + \sum_{j\in\mathcal{D}} G_-(i,j)d_j.
\end{aligned}$$

Proof. As in the proof of Proposition 2.2, let Y_i, $i \in \mathcal{D}$, be random variables independent of the sequences $(X_n)_{n\in\mathbb{Z}}$ and $(Q_{1,i}, M_{1,i})_{i\in\mathcal{D}}$, such that $P(Y_i > t) = P(Y > t|X_0 = i)$ and $P(Y_i < -t) = P(Y < -t|X_0 = i)$ for all $t > 0$. It follows from (17) that

$$P(Q_1 + M_1 Y > t|X_1 = i) = \sum_{j\in\mathcal{D}} P(Q_{1,i} + M_{1,i}Y_j > t)H(i,j).$$

Similarly,

$$P(Q_1 + M_1 Y < -t|X_1 = i) = \sum_{j\in\mathcal{D}} P(Q_{1,i} + M_{1,i}Y_j < -t)H(i,j).$$

This yields the result of the lemma by applying Lemma 4 of [12] separately to each of the terms $P(Q_{1,i} + M_{1,i}Y_j > t)$ and $P(Q_{1,i} + M_{1,i}Y_j < -t)$. □

We are now in a position to complete the proof of Theorem 1.4. The argument is essentially the same as the one employed in [12] for the i.i.d. coefficients (Q_n, M_n). Define

$$\begin{aligned} C_+(i) &:= \limsup_{t\to\infty} \frac{P(R > t|X_0 = i)}{t^{-\alpha}L(t)}, \qquad C_-(i) := \limsup_{t\to\infty} \frac{P(R < -t|X_0 = i)}{t^{-\alpha}L(t)}, \\ D_+(i) &:= \liminf_{t\to\infty} \frac{P(R > t|X_0 = i)}{t^{-\alpha}L(t)}, \qquad D_-(i) := \liminf_{t\to\infty} \frac{P(R < -t|X_0 = i)}{t^{-\alpha}L(t)}. \end{aligned}$$

Let R^* be a stationary solution of the equation $R_{n+1} = |Q_n| + |M_n|R_n$. Note that $-R^*$ is a stationary solution of the equation $R_{n+1} = -|Q_n| + |M_n|R_n$.

Since, for $t > 0$,

$$P(-R^* > t) \le P(R > t) \le P(R^* > t) \text{ and } P(R^* < -t) \le P(R < -t) \le P(-R^* < -t),$$

Theorem 1.3 implies

$$\begin{aligned} -\infty < \lim_{t\to\infty} \frac{P(R^* < -t|X_0 = i)}{t^{-\alpha}L(t)} \le D_+ \le C_+ \le \lim_{t\to\infty} \frac{P(R^* > t|X_0 = i)}{t^{-\alpha}L(t)} < +\infty, \\ -\infty < \lim_{t\to\infty} \frac{P(R^* < -t|X_0 = i)}{t^{-\alpha}L(t)} \le D_- \le C_- \le \lim_{t\to\infty} \frac{P(R^* > t|X_0 = i)}{t^{-\alpha}L(t)} < +\infty. \end{aligned}$$

Therefore, one can apply Lemma 3.1 to vectors $C_\pm$ and $D_\pm$ defined as $C_\pm := \big(C_\pm(i)\big)_{i\in\mathcal{D}}$ and $D_\pm := \big(D_\pm(i)\big)_{i\in\mathcal{D}}$, to obtain

$$C_+ + C_- \le q + G(C_+ + C_-) \quad \text{and} \quad D_+ + D_- \ge q + G(D_+ + D_-),$$

which implies

$$(I - G)^{-1}q \le D_+ + D_- \le C_+ + C_- \le (I - G)^{-1}q.$$

This is only possible if

$$C_+ = D_+ = \lim_{t\to\infty} \frac{P(R > t|X_0 = i)}{t^{-\alpha}L(t)} \quad \text{and} \quad C_- = D_- = \lim_{t\to\infty} \frac{P(R < -t|X_0 = i)}{t^{-\alpha}L(t)},$$

and the inequalities in the conclusions of Lemma 3.1 are actually equalities. The latter implication yields

$$\begin{cases} C_+ = q_+ + G_+C_+ + G_-C_- \\ C_- = q_- + G_+C_- + G_-C_+ \end{cases},$$

implying the result of Theorem 1.4.

4 Proof of Theorem 1.7

First, we define a sequence of regeneration times for the **backward** chain $Z_n = Y_{-n}$, $n \in \mathbb{Z}$, where the Markov chain $(Y_n)_{n\in\mathbb{Z}}$ is introduced in (7) and in Theorem A. Let $K(x,y)$, $x,y \in \mathcal{S}$ be the transition kernel of Y_n and denote $\pi(x) = P(Y_n = x)$, $x \in \mathcal{S}$. Then transition kernel H of Z_n is given by

$$H(x,y) = P(Y_n = y | Y_{n+1} = x) = \frac{\pi(y)}{\pi(x)} K(y,x). \tag{31}$$

In particular, Z_n is an irreducible and has the same stationary distribution $\pi = \big(\pi(x)\big)_{x\in\mathcal{S}}$ as Y_n.

Fix a state $y^* \in \mathcal{S}$ and a constant $r \in (0,1)$. Let $(\eta_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. random variables independent of both $(Z_n)_{n\in\mathbb{Z}}$ and the induced process $\big(\widetilde{Q}_{n,y}, \widetilde{M}_{n,y}\big)_{n\in\mathbb{Z}, y\in\mathcal{S}}$ defined in (12), such that $P(\eta_0 = 1) = r$ and $P(\eta_0 = 0) = 1 - r$. Define $N_0 = 0$, $N_i = \inf\{n > N_{i-1} : Z_n = y^*, \eta_n = 1\}$, $i \in \mathbb{N}$. Then the blocks $(Z_{N_i}, \ldots, Z_{N_{i+1}-1})$ are independent for $i \geq 0$ and identically distributed for $i \geq 1$. Intuitively, each time when the chain Z_n arrives to a distinguished state y^*, an independent coin with chances for "Head" (="success") and "Tail" (="failure") equal to respectively r and $1-r$ is flipped. If the coin shows "success", the regeneration event is declared.

Note that between two successive regeneration times, the chain $(Z_n)_{n\geq 0}$ evolves according to a sub-stochastic Markov kernel Θ defined by

$$H(x,y) = \Theta(x,y) + r\mathbf{1}_{\{y=y^*\}} H(x,y), \qquad x,y \in \mathcal{S}. \tag{32}$$

That is

$$\Theta(x,y) = P(Z_1 = y, N_1 > 1 | Z_0 = x). \tag{33}$$

We remark that the properties of the Markov chain $(Y_n)_{n\in\mathbb{Z}}$ listed in the statement of Lalley's Theorem A imply that $E\big(e^{\beta N_1} \big| Z_0\big)$ is uniformly bounded for some $\beta > 0$ (see the paragraph right after the statement of Theorem 1 in [20]).

By using the regeneration times $(N_i)_{i\geq 0}$ we can associate (1) with an equivalent linear recursion model having i.i.d. coefficients. More precisely, for $i \geq 0$, let

$$\begin{aligned} A_i &= Q_{N_i} + Q_{N_i+1} M_{N_i} + \ldots + Q_{N_{i+1}-1} M_{N_i} M_{N_i+1} \ldots M_{N_{i+1}-2} && (34)\\ B_i &= M_{N_i} M_{N_i+1} \ldots M_{N_{i+1}-1} && (35) \end{aligned}$$

The pairs (A_i, B_i) are independent for $i \geq 0$, identically distributed for $i \geq 1$, and

$$R = A_0 + \sum_{n=1}^{\infty} A_n \prod_{i=0}^{n-1} B_i. \tag{36}$$

The idea of the proof of Theorem 1.7 is to show that the conditions of the following theorem are satisfied for the random variables (A_i, B_i), $i \geq 1$.

Theorem B (Kesten [19], Goldie [10])**.** *Let $(A_i, B_i)_{i\geq 1}$ be an i.i.d. sequence and* *(i) For some $\alpha > 0$, $E\left(|A_1|^\alpha\right) = 1$, $E\left(|B_1|^\alpha \log^+ |B_1|\right) < \infty$, and $E(|A_1|^\alpha) < \infty$.*

(ii) $P(\log |B_1| \in \delta \cdot \mathbb{Z} | B_1 \neq 0) < 1$ for all $\delta > 0$.

Let

$$\widetilde{R} = A_1 + \sum_{n=2}^{\infty} A_n \prod_{i=1}^{n-1} B_i. \tag{37}$$

Then

(a) $\lim_{t\to\infty} t^\alpha P\big(\widetilde{R} > t\big) = K_+$ and $\lim_{t\to\infty} t^\alpha P\big(\widetilde{R} < -t\big) = K_-$ for some $K_+ \geq 0$ and $K_- \geq 0$.

(b) If $P(B_1 < 0) > 0$ then $K_+ = K_-$.

(c) $K_+ + K_- > 0$ if and only if $P\big(A_1 = (1 - B_1)c\big) < 1$ for all $c \in \mathbb{R}$.

The conditions of Theorem B for the random pair (A_1, B_1) are verified in the following series of lemmas. First, we observe that Lalley's Theorem A and (31) imply that the Markov chain $(Z_n, Q_{-n}, M_{-n})_{n\in\mathbb{Z}}$ satisfies Assumption 1.2 in [26]. Therefore, Lemma 2.3 along with Proposition 2.4 in [26] yield the following Perron-Frobenius type result. For $\beta \geq 0$, let

$$H_\beta(x,y) = H(x,y) E\big(|M_0|^\beta \big| Z_0 = y\big), \quad \Theta_\beta(x,y) = \Theta(x,y) E\big(|M_0|^\beta \big| Z_0 = y\big).$$

Notice that

$$\begin{aligned} &E\Big(\prod\nolimits_{i=0}^n |M_{-i}|^\beta \mathbf{1}_{\{Z_n = y\}} \Big| Z_0 = x\Big) = E\big(|M_0|^\beta \big| Z_0 = x\big) H_\beta^n(x,y) \\ &E\Big(\mathbf{1}_{\{n<N_1\}} \prod\nolimits_{i=0}^n |M_{-i}|^\beta \mathbf{1}_{\{Z_n = y\}} \Big| Z_0 = x\Big) = E\big(|M_0|^\beta \big| Z_0 = x\big) \Theta_\beta^n(x,y), \qquad y \neq y^*. \end{aligned}$$

Let $\mathbf{M}_b = \{f = \big(f(z)\big)_{z\in\mathcal{S}} : f(z) \in \mathbb{R}$ for all $z \in \mathcal{S}$ and $\sup_{z\in\mathcal{S}} |f_z| < +\infty\}$ be the space of bounded elements of $\mathbb{R}^{\mathcal{S}}$. In the next proposition we consider matrices of the form $A(x,y)$, $x, y \in \mathcal{S}$, as operators acting in $\mathbf{M}_b$.

Proposition 4.1 (See Lemma 2.3 and Proposition 2.4 in [26])**.**

(a) For any $\beta > 0$ and every $x \in \mathcal{S}$, the following limit exists and does not depend on x :

$$\Lambda(\beta) := \lim_{n\to\infty} \frac{1}{n} \log E\Big(\prod\nolimits_{i=0}^{n-1} |M_i|^\beta \Big| Z_0 = x\Big).$$

Moreover, for some constants $c_\beta \geq 1$ that depend on β only,

$$c_\beta^{-1} e^{n\Lambda(\beta)} \leq E\Big(\prod\nolimits_{i=0}^{n-1} |M_i|^\beta \Big| Z_0 = x\Big) \leq c_\beta e^{n\Lambda(\beta)}, \quad \forall\, n \in \mathbb{N},\ P - a.s.$$

(b) There exists a unique $\alpha > 0$ such that $\Lambda(\alpha) = 0$, $\Lambda(\beta)(\beta - \alpha) > 0$ for all $\beta > 0$, $\beta \neq \alpha$.

(c) For any $\beta \geq 0$, there exists a vector $f^{(\beta)} = \big(f^{(\beta)}(z)\big)_{z\in\mathcal{S}} \in \mathbf{M}_b$ such that $\inf_{z\in\mathcal{S}} f^{(\beta)}(z) > 0$ and $H_\beta f^{(\beta)} = r_\beta f^{(\beta)}$, where r_β is the spectral radius of the matrix H_β.

(d) For any $\beta \geq 0$, there exists a vector $g^{(\beta)} = \big(g^{(\beta)}(z)\big)_{z\in\mathcal{S}} \in \mathbf{M}_b$ such that $\inf_{z\in\mathcal{S}} g^{(\beta)}(z) > 0$ and $\Theta_\beta g^{(\beta)} = \rho_\beta g^{(\beta)}$, where ρ_β is the spectral radius of the matrix Θ_β.

(e) *For any* $\beta \geq 0$, *we have* $\rho_\beta \in (0, r_\beta)$.

Notice that the above proposition is the only statement in this section that is borrowed as is from [26], and thus is the only claim that makes the proof of Theorem 1.7 given here not entirely self-contained.

We will next show that $E\big(|B_1|^\alpha\big) = 1$.

Lemma 4.2. $E(|B_1|^\alpha) = 1$.

Proof. By virtue of Proposition 4.1 the spectral radius r_α of H_α is one, while the spectral radius ρ_α of Θ_α is strictly less than one. It follows from (32) that the vector $f^{(\alpha)}$ normalized by the condition $f^{(\alpha)}(y^*) = 1$ is the unique positive vector in $\mathbf{M}_b$, solving the equation $(I - \Theta_\alpha)f = s$, where

$$s(x) := rH(x, y^*) \cdot E\big(|M_0|^\alpha \big| Z_0 = y^*\big).$$

Hence (this is a very particular case of the results of [1] and [23, Theorem 5.1])

$$f^{(\alpha)}(x) = \sum_{n=0}^{\infty} \Theta_\alpha^n s(x) = E\Big(\prod\nolimits_{i=1}^{N_1} |M_{-i}|^\alpha \Big| Z_0 = x\Big), \tag{38}$$

and

$$E\Big(\prod\nolimits_{i=1}^{N_1} |M_{-i}|^\alpha \Big| Z_0 = y^*\Big) = E\Big(\prod\nolimits_{i=0}^{N_1-1} |M_{-i}|^\alpha \Big| Z_0 = y^*\Big) = E(|B_1|^\alpha) = 1. \tag{39}$$

The second equality in (38) follows since the chain (Z_i) evolves according to the kernel Θ until N_1 (see (33)), while (39) follows from the normalization condition $f^{(\alpha)}(y^*) = 1$. □

We will next show that condition (iv) of Theorem 1.7 implies (actually it is equivalent to the fact) that $\log|B_0|$ is non-lattice.

Lemma 4.3. *Assume that* $y^* \in \mathcal{D}^1$ *and let* $V = \sum\limits_{n=0}^{N_1-1} \log|M_{-n}| = \log|B_0|$. *Then for any* $\delta > 0$ *we have* $P\big(V \in \delta \cdot \mathbb{Z} \big| Z_0 = y^*\big) < 1$

Proof. For $y \in \mathcal{D}$ and $k \in \mathbb{N}$, let $y_{(k)}$ denote $(y_1, \ldots, y_k) \in \mathcal{D}^k$ with $y_2 = \cdots = y_k = y$ and $y_1 = y^*$. By Lalley's Theorem A, for any $y \in \mathcal{D}$, we have

$$P(Z_0 = y_{(1)}, Z_1 = y_{(m)}, \ldots, Z_{m-1} = Z_{N_1-1} = y_{(2)}) > 0 \tag{40}$$

for some $m > 1$. Therefore, $P\big(V \in \delta \cdot \mathbb{Z} | Z_0 = (y^*)\big) < 1$ along with (40) would imply: (i) using first $y = y^*$, that $P\big(m \cdot \log|M_{0,\zeta(y^*)}| \in \delta \cdot \mathbb{Z} | Z_0 = (y^*)\big) = 1$; (ii) and then, using a general $y \in \mathcal{D}$, that $P\big((m-1) \cdot \log|M_{0,\zeta(y)}| + \log|M_{0,\zeta(y^*)}| \in \delta \cdot \mathbb{Z} \big| Z_0 = (y^*)\big) = 1$. Therefore, we would have $P\big(m \cdot \log|M_{0,\zeta(y)}| \in \delta \cdot \mathbb{Z} \big| Z_0 = (y^*)\big) = 1$ for any $y \in \mathcal{D}$ contradicting condition (iv) of Theorem 1.7. The proof of the lemma is completed. □

To show that the conditions of Theorem B are satisfied for the sequence of random pairs $(A_n, B_n)_{n \in \mathbb{Z}}$, it remains to show that

$$E\big(|A_0|^\alpha\big) < \infty \quad \text{and} \quad E\big(|B_0|^\alpha \log^+|B_0|\big) < \infty. \tag{41}$$

Thus, in virtue of condition (i) of Theorem 1.7, it suffices to prove the following lemma.

Lemma 4.4. *There exists $\beta > \alpha$ such that*

$$A_\beta(x) := E\Big[\Big(\sum_{n=0}^{N_1-1}\prod_{i=0}^{n-1}|M_{-i}|\Big)^\beta\Big|Z_0 = x\Big] \text{ is a bounded function of } x \in \mathcal{S}. \tag{42}$$

Proof. Since for any $n \in \mathbb{N}$ and positive numbers $\{a_i\}_{i=1}^n$ we have

$$(a_1 + a_2 + \dots a_n)^\beta \le n^\beta(a_1^\beta + a_2^\beta + \dots a_n^\beta),$$

we obtain for any $\beta > 0$ and $x \in \mathcal{S}$:

$$\begin{aligned} A_\beta(x) &= E\Big[\Big(\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\prod_{j=0}^{i-1}|M_{-j}|\mathbf{1}_{\{N_1=n\}}\Big)^\beta\Big|Z_0 = x\Big] \\ &= \sum_{n=1}^{\infty}E\Big[\Big(\sum_{i=0}^{n-1}\prod_{j=0}^{i-1}|M_{-j}|\mathbf{1}_{\{N_1=n\}}\Big)^\beta\Big|Z_0 = x\Big] \\ &\le \sum_{n=1}^{\infty}n^\beta\sum_{i=0}^{n-1}E\Big(\prod_{j=0}^{i-1}|M_{-j}|^\beta\mathbf{1}_{\{N_1\ge n\}}\Big|Z_0 = x\Big). \end{aligned} \tag{43}$$

But $E\Big(\prod_{j=0}^{i-1}|M_{-j}|^\beta\mathbf{1}_{\{N_1\ge n\}}\Big|Z_0 = x\Big) = E\big(|M_0|^\beta\big|Z_0 = x\big)\cdot\Theta^{n-i+1}\Theta_\beta^{i-1}\mathbf{1}(x)$. Since the spectral radius of both the matrices Θ_α and $\Theta = \Theta_0$ are strictly less than one by Proposition 4.1, it follows from (43) that (42) holds for some $\beta > \alpha$. This yields (41). □

Completion of the proof of Theorem 1.7
(a)–(d)
It follows from Lemmas 4.2–4.4 that the conclusions of Theorem B can be applied to the random pair (A_1, B_1) defined in (34) and (35). Claims (a) through (d) of Theorem 1.7 follow then from Proposition 2.2 applied to $R = A_0 + B_0\widetilde{R}$ and the conditional measures $P(\cdot|Z_0 = x)$. In particular,

$$\begin{aligned} \lim_{t\to\infty}P\big(R > t|Z_0 = x\big) &= E\big((B_0\mathbf{1}_{\{B_0>0\}})^\alpha\big|Z_0 = x\big)\cdot K_+ \\ &+ E\big((B_0\mathbf{1}_{\{B_0<0\}})^\alpha\big|Z_0 = x\big)\cdot K_-, \end{aligned} \tag{44}$$

and

$$\begin{aligned} \lim_{t\to\infty}P\big(R < -t|Z_0 = x\big) &= E\big((B_0\mathbf{1}_{\{B_0<0\}})^\alpha\big|Z_0 = x\big)\cdot K_+ \\ &+ E\big((B_0\mathbf{1}_{\{B_0>0\}})^\alpha\big|Z_0 = x\big)\cdot K_-, \end{aligned} \tag{45}$$

where the constants K_+ and K_- are defined in the statement of Theorem B. Notice that conclusion (v) of Lalley's Theorem A imply that $P(N_1 \text{ is odd}) > 0$ and $P(N_1 \text{ is even}) > 0$. Therefore, in virtue of (35), $P(M_0 < 0) > 0$ implies $P(B_0 < 0|Z_0 = x) > 0$ for all $x \in \mathcal{S}$.

(e)

First, notice that (13) implies that if $R_0 = \Gamma(X_0)$ then $P\big(R_n = \Gamma(X_n)\big) = 1$. Hence if $R_0 = \Gamma(X_0)$, then, for any $n \in \mathbb{N}$, R_n can take only a finite number of values belonging to the set $\{\Gamma(i) : i \in \mathcal{D}\}$, in which case the sequence R_n cannot converge in distribution to a heavy-tailed random variable. Therefore, since the limiting random variable R is independent of the choice of R_0, (13) implies $P\big(K_1(\mathcal{X}) + K_{-1}(\mathcal{X}) = 0\big) = 1$.

On the other hand, (44), (45), and condition (ii) of Theorem 1.7 combined together imply that $P\big(K_1(\mathcal{X}) + K_{-1}(\mathcal{X}) = 0\big)= 1$ if and only if $K_+ = K_- = 0$. Furthermore, according to Theorem B, $K_+ = K_- = 0$ if and only if $P\big(A_1 + c(y^*)B_1 = c(y^*)\big)$ for some $c(y^*) \in \mathbb{R}$, in which case $P\big(\widetilde{R} = c(y^*)\big) = 1$.

Let $\widehat{R} = \frac{R-Q_0}{M_0}$. Then, $P\big(\widetilde{R} = c(y^*)\big) = P\big(R = c(y^*)\big|Z_0 = y^*\big) = 1$ implies that

$$P\big(R = c(y^*)\big|Z_0 = y^*\big) = P\big(Q_0 + M_0\widehat{R} = c(y^*)\big|Z_0 = y^*\big) = 1.$$

Since $\widehat{R}$ is conditionally independent of (Q_0, M_0) given Z_0, it follows that for some $c_1(y^*) \in \mathbb{R}$,

$$P\big(\widehat{R} = c_1(y^*)\big|Z_0 = y^*\big) = P\big(R = c_1(y^*)\big|Z_{-1} = y^*\big) = 1 \tag{46}$$

and

$$P\big(\widetilde{Q}_{0,\zeta(y^*)} + \widetilde{M}_{0,\zeta(y^*)}c_1(y^*) = c(y^*)\big) = 1. \tag{47}$$

It follows from (46) and the Markov property that

$$P\big(R = c_1(y^*)\big|Z_0 = y\big) = 1 \text{ for any } y \in \mathcal{S} \text{ such that } P(Z_0 = y|Z_{-1} = y^*) > 0,$$

and hence

$$P\big(R = c(y^*)\big|Z_{-1} = x\big) = 1 \text{ for any } x \in \mathcal{S} \text{ such that } P(Z_0 = y^*|Z_{-1} = x) > 0. \tag{48}$$

It follows from (46) and (48) that

$$P\big(R = c_1(x) = c(y^*)\big|Z_{-1} = x\big) = 1 \ \forall\ x \in \mathcal{S} \text{ such that } P(Z_0 = y^*|Z_{-1} = x) > 0.$$

Since $y^* \in \mathcal{S}$ is an arbitrary state, we obtain from (47) that

$$P\big(\widetilde{Q}_{0,\zeta(Z_0)} + \widetilde{M}_{0,\zeta(Z_0)}c_1(Z_0) = c_1(Z_{-1})\big) = 1,$$

which is (13) once one set $\Gamma(Y_n) = c_1(Z_{-n})$.

Assume now that $(Q_{0,i}, M_{0,i})$ is non-degenerate for any $i \in \mathcal{D}$. Since the following system of equation for two unknown variables q, m,

$$\begin{cases} q + mc_1 = c \\ q + m\tilde{c}_1 = \tilde{c} \end{cases},$$

has a unique solution unless $c_1 = \tilde{c}_1$, it follows from (47) that c_1 depends on $\zeta(y^*)$ only. Define function $\Gamma : \mathcal{D} \to \mathbb{R}$ by setting $\Gamma(i) = c_1(y^*)$ for $i = \zeta(y^*)$. Then, (47) along with (48) imply $P\big(Q_0 + M_0\Gamma(X_0) = \Gamma(X_1)\big) = 1$. But then (8) implies that $\Gamma(X_n)$ is a constant function of X_n, that is $P\big(Q_0 + M_0c = c\big) = 1$ for some $c \in \mathbb{R}$, as required.

The proof of Theorem 1.7 is completed.

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