Discrete-time Ornstein-Uhlenbeck process in a stationary dynamic environment

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Abstract

We study the stationary solution to the recursion $X_{n+1} = \gamma X_n + \xi_n$, where $\gamma \in (0,1)$ is a constant and ξ_n are Gaussian variables with *random* parameters. Specifically, we assume that $\xi_n = \mu_n + \sigma_n \varepsilon_n$, where $(\varepsilon)_{n\in\mathbb{Z}}$ is an i.i.d. sequence of standard normal variables and $(\mu_n, \sigma_n)_{n\in\mathbb{Z}}$ is a stationary and ergodic process independent of $(\varepsilon_n)_{n\in\mathbb{Z}}$, which serves as an exogenous dynamic environment for the model.

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1 Introduction

This paper is devoted to the study of solutions to the following linear recursion (*stochastic difference equation*):

$$X_{n+1} = \gamma X_n + \xi_n, \tag{1}$$

where $\gamma \in (0,1)$ is a constant and $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary and ergodic sequence of normal variables with *random* means and variances. More precisely, we suppose that

$$\xi_n = \mu_n + \sigma_n \varepsilon_n, \qquad n \in \mathbb{Z},$$

where $(\varepsilon_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence of standard (zero mean and variance one) Gaussian random variables and $(\mu_n, \sigma_n)_{n\in\mathbb{Z}}$ is an independent of it stationary and ergodic process. Denote

$$\omega_n = (\mu_n, \sigma_n) \in \mathbb{R}^2, \qquad n \in \mathbb{Z}, \tag{2}$$

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and $\omega = (\omega_n)_{n\in\mathbb{Z}}$. We refer to the sequence ω as a *random dynamic environment* or simply *dynamic environment*. We denote the probability law of the random environment ω by P and denote the corresponding expectation operator by E_P. Throughout the paper we impose the following conditions on the coefficients in (1).

Assumption 1.1. *Assume that:*

(A1) The sequence of pairs $(\omega_n)_{n\in\mathbb{Z}}$ *is stationary and ergodic.*

(A2) $E_P\big(\log^+|\mu_0| + \log^+|\sigma_0|\big) < +\infty$, *where* $x^+ := \max\{x, 0\}$ *for* $x \in \mathbb{R}$.

(A3) $\gamma \in (0, 1)$ *is a constant.*

The conditions stated in Assumption 1.1 ensure the existence of a limiting distribution for X_n and, consequently, the existence of a (unique) stationary solution to (1) (see Theorem 2.2 below). This solution is very well understood in the classical case when the innovations ξ_n form an i.i.d. sequence, in which case $(X_n)_{n\geq 0}$ defined by (1) is an *AR(1) (first-order autoregressive) process*. The AR(1) process often serves to model discrete-time dynamics of both the value as well as the volatility of financial assets and interest rates, see for instance, [31, 33].

A sequence X_n that solves (1) can be thought as the AR(1) process with *stochastic variance*. The recognition that financial time-series, such as stock returns and exchange rates, exhibit changes in volatility over time goes back at least to [15]. These changes are due for example to seasonal effects, response to the news and dynamics of the market. In this context, random environment ω can be interpreted as an exogenous to the model factor determined by the current state of the underlying economy. For a comparative review of stochastic variance models we refer the reader to [9, 32, 33].

When ω_n is a function of the state of a Markov chain, stochastic difference equation (1) is a formal analogue of the Langevin equation with regime switches, which was studied in [14]. The notion of *regime shifts* or *regime switches* traces back to [19], where it was proposed in order to explain the cyclical feature of certain macroeconomic variables. Discrete-time linear recursions with Markov-dependent coefficients have been considered, for instance, in [2, 3, 10, 11, 29]. Certain stationary non-Markovian sequences of coefficients ξ_n in (1) were considered, for instance, in [5, 18, 27]. To enable some explicit computation in a few examples in this paper we will consider the following setup.

Assumption 1.2. *Let* $(y_n)_{n\in\mathbb{Z}}$ *be an irreducible Markov chain defined on a finite state space* $\mathcal{D} = \{1, \ldots, d\}$, $d \in \mathbb{N}$, *and suppose that the sequence* $(\xi_n)_{n\in\mathbb{Z}}$ *is induced (modulated) by* $(y_n)_{n\in\mathbb{Z}}$ *as follows. Assume that for each* $i \in \mathcal{D}$ *there exists an i.i.d. sequence of pairs of reals* $\omega_i = (\mu_{i,n}, \sigma_{i,n})_{n\in\mathbb{Z}}$ *and that these sequence are independent each of other. Further, suppose that (A2) of Assumption 1.1 holds for each* $i \in \mathcal{D}$, *with* (μ_0, σ_0) *replaced by* $(\mu_{i,0}, \sigma_{i,0})$. *Finally, define* $\mu_n = \mu_{y_n,n}$ *and* $\sigma_n = \sigma_{y_n,n}$.

Equation (1) with i.i.d. but not necessarily Gaussian coefficients $(\xi_n)_{n\in\mathbb{Z}}$ has been considered, for example, in [1, 5, 16, 22, 23, 24, 25, 28, 36], see also references therein. The stationary solution to (1) is often referred to as a *discrete-time (generalized) Ornstein-Uhlenbeck process*. We adopt here a similar terminology, and call the above model, *discrete-time Ornstein-Uhlenbeck process in a stationary dynamic environment*. The case when γ

is close to one is often of a special interest in the context of stochastic volatility models; see [33, Section 3.5]. Such *nearly unstable* processes have been considered, for instance, in [6, 8, 22, 23, 26].

In this work we study the probabilistic structure of the (unique) stationary solution to (1) under general the Assumption 1.1. The rest of the paper is organized as follows. Section 2 is devoted to the study of (Gaussian) sequence $(X_n)_{n\in\mathbb{N}}$ in a fixed environment. In Section 3 we study the asymptotic behavior of the limiting distribution of X_n, as $\gamma \to 1^-$. Section 4 contains a limit theorem for the extreme values $M_n = \max_{1\le k\le n} X_k$ and a related to it law of iterated logarithm. In Section 5 we investigate the asymptotic behavior of the partial sums $S_n = \sum_{k=1}^n X_k$.

2 Stationary distribution of the sequence $(X_n)_{n\in\mathbb{N}}$

We denote the conditional law of $(\xi_n)_{n\in\mathbb{Z}}$, given an environment ω, by P_ω and the corresponding expectation by E_ω. To emphasize the existence of two levels of randomness in the model, the first one due to the random environment and the second one due to the randomness of $(\varepsilon_n)_{n\in\mathbb{N}}$, we will use the notations $\mathbb{P}$ and $\mathbb{E}$ for the unconditional distribution of $(\xi_n)_{n\in\mathbb{Z}}$ (and $(X_n)_{n\in\mathbb{Z}}$) and the corresponding expectation operator, respectively. We thus have

$$\mathbb{P}(\,\cdot\,) = \int P_\omega(\,\cdot\,)dP(\omega) = E_P\big[P_\omega(\,\cdot\,)\big]. \tag{3}$$

For any constants $\mu \in \mathbb{R}$ and $\sigma > 0$, we denote by Φ_{μ,σ^2} the distribution function of a normal random variable with mean μ and variance σ^2. That is, for $t \in \mathbb{R}$,

$$G_{\mu,\sigma^2}(t) := 1 - \Phi_{\mu,\sigma^2}(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_t^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx. \tag{4}$$

It will be notationally convenient to extend the notion of "normal variables" to a class of distributions with random parameters μ and σ.

Definition 2.1. Let (μ, σ) be a random $\mathbb{R}^2$-valued vector with $\mathbb{P}(\sigma > 0) = 1$. We say that a random variable X has $\mathcal{N}(\mu, \sigma^2)$-distribution (in words, *normal* (μ, σ^2) *distribution*) and write $X \sim \mathcal{N}(\mu, \sigma^2)$ if

$$\mathbb{P}(X \le t) = \mathbb{E}\big[\Phi_{\mu,\sigma^2}(t)\big], \qquad t \in \mathbb{R}.$$

That is, conditional on (μ, σ^2) distribution of X is normal with mean μ and variance σ^2.

2.1 Limiting distribution of X_n

First we discuss (marginal) distribution of in individual member of the sequence X_n. It follows from (1) that

$$X_n = \gamma^n X_0 + \sum_{t=1}^{n-1} \gamma^{n-t-1} \xi_t. \tag{5}$$

The following general result can be deduced from (5) (see for instance [4]):

Proposition 2.2. *Assume that*

(i) $(\xi_n)_{n\in\mathbb{Z}}$ *is a stationary and ergodic sequence.*

(ii) $\mathbb{E}\big[\log^+|\xi_0|\big] < +\infty$*, where* $x^+ := \max\{x, 0\}$ *for* $x \in \mathbb{R}$*.*

(iii) $\gamma \in (0, 1)$ *is a constant.*

Then, for any initial value X_0*, the series* X_n *defined by* (1) *converges in distribution, as* $n \to \infty$*, to the random variable*

$$X = \sum_{t=0}^{\infty} \gamma^t \xi_{-t}, \tag{6}$$

which is the unique initial value making $(X_n)_{n\geq 0}$ *into a stationary sequence. Furthermore, the series in the right-hand side of* (6) *converges absolutely with probability one.*

Our first result is a characterization of X as a mixture of Gaussian random variables under Assumption 1.1. Let

$$\theta = \sum_{k=0}^{\infty} \gamma^k \mu_{-k} \qquad \text{and} \qquad \tau = \Big(\sum_{k=0}^{\infty} \gamma^{2k} \sigma_{-k}^2\Big)^{1/2}. \tag{7}$$

Notice that by Theorem 2.2, the random variables θ and τ are well-defined functions of the environment. Recall Definition 2.1.

We have:

Theorem 2.3. *Let Assumption 1.1 hold and let* X *be defined by* (6)*. Then* $X \sim \mathcal{N}(\theta, \tau^2)$*.*

Proof. It follows from (5) and (6) that the stationary solution is given by

$$X_n = \sum_{k=0}^{\infty} \gamma^k \xi_{n-k} = \gamma^n \sum_{j=-\infty}^{n} \gamma^{-j} \xi_j. \tag{8}$$

It suffices to show that

$$\lim_{n\to\infty} \mathbb{E}\big[e^{itX_n}\big] = E_P\Big[e^{i\theta t - \frac{\tau^2 t^2}{2}}\Big], \qquad t \in \mathbb{R}.$$

Since environment ω is a stationary sequence, (8) implies that $\mathbb{E}\big[e^{itX_n}\big] = \mathbb{E}\big[e^{itY_n}\big]$ where $Y_n \sim \mathcal{N}(\theta_n, \tau_n^2)$ with τ_n and θ_n given by

$$\theta_n = \sum_{k=0}^{n} \gamma^k \mu_{-k} \qquad \text{and} \qquad \tau_n = \Big(\sum_{k=0}^{n} \gamma^{2k} \sigma_{-k}^2\Big)^{1/2}.$$

It follows from (7) and Theorem 2.2 that

$$\lim_{n\to\infty} \theta_n = \theta \qquad \text{and} \qquad \lim_{n\to\infty} \tau_n = \tau, \quad P-\text{a.s.}$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\big[e^{itX_n}\big] &= \lim_{n\to\infty}\mathbb{E}\big[e^{itY_n}\big]=\lim_{n\to\infty}E_P\Big[E_\omega\big[e^{itY_n}\big]\Big]\\ &= \lim_{n\to\infty}E_P\Big[e^{i\theta_n t-\frac{\tau_n^2t^2}{2}}\Big]=E_P\Big[\lim_{n\to\infty}e^{i\theta_n t-\frac{\tau_n^2t^2}{2}}\Big]=E_P\Big[e^{i\theta t-\frac{\tau^2t^2}{2}}\Big].\end{aligned}$$

To justify interchanging of the limit and expectation operator in the last but one step, observe that $\left|e^{i\theta_n t-\tau_n^2t^2}\right|\le 1$, and therefore the bounded convergence theorem can be applied. □

Corollary 2.4. *Let Assumption 1.1 hold. Then the distribution of X is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. By Fubini's theorem, for any Borel set $A\subset\mathbb{R}$,

$$\begin{aligned}\mathbb{P}(X\in A)=\mathbb{E}\big[\mathcal{N}(\theta,\tau^2)\in A\big] &= \int\left(\frac{1}{\sqrt{2\pi\tau^2}}\int_A e^{-\frac{(x-\theta)^2}{2\tau^2}}dx\right)dP(\omega)\\ &= \int_A\left(\int\frac{1}{\sqrt{2\pi\tau^2}}e^{-\frac{(x-\theta)^2}{2\tau^2}}dP(\omega)\right)dx,\end{aligned}$$

and, furthermore, the integral $\int\frac{1}{\sqrt{2\pi\tau^2}}e^{-\frac{(x-\theta)^2}{2\tau^2}}P(d\omega)$ exists for m-a.e. x, where m denotes the Lebesgue measure of the Borel subsets of $\mathbb{R}$. □

Corollary 2.5. *Let Assumption 1.1 hold. Then $\mathbb{E}\big[|X-\theta|^p\big]=\frac{2^{\frac{p}{2}}\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}\cdot E_P[\tau^p]$ for any $p>-1$.*

2.2 Distribution tails of X

The next theorem shows that under a mild extra assumption, the tails of the distribution of X have asymptotically Gaussian structure. For a random variable Y denote

$$\|Y\|_p=\Big(\mathbb{E}\big[|Y|^p\big]\Big)^{1/p}\quad(p\ge 1)\quad\text{and}\quad\|Y\|_\infty=\inf\big\{y\in\mathbb{R}:\mathbb{P}\big(|Y|>y\big)=0\big\}.$$

Recall (see, for instance, [13, p. 466]) that $\|Y\|_\infty=\lim_{p\to\infty}\|Y\|_p$. Notice that (7) implies $\|\tau\|_\infty^2\le(1-\gamma^2)^{-1}\cdot\|\sigma_0\|_\infty^2$.

We have:

Theorem 2.6. *Let Assumption 1.1 and assume in addition that $P\big(|\mu_0|+\sigma_0<\lambda\big)=1$ for some constant $\lambda>0$. Then*

$$\lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{P}(X>t)=\lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{P}(X<-t)=-\frac{1-\gamma^2}{2\Lambda^2},$$

where $\Lambda:=\|\tau\|_\infty\in(0,\infty)$.

Proof. The proof relies on some well-known bounds for the tails of normal distributions. For the reader's convenience we will give a short derivation of these bounds here. We will only consider the upper tails $\mathbb{P}(X > t)$. The lower tails $\mathbb{P}(X < -t)$ can be treated in the same manner exactly, and therefore the proof for the lower tails is omitted.

Recall $G_{\mu,\sigma^2}(t)$ from (4). We have:

$$\begin{aligned} G_{0,\sigma^2}(t) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_t^\infty e^{-\frac{x^2}{2\sigma^2}} dx \le \frac{1}{\sqrt{2\pi\sigma^2}} \int_t^\infty \frac{x}{t} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{2\sqrt{2\pi\sigma^2 t^2}} \int_{t^2}^\infty e^{-\frac{y}{2\sigma^2}} dy \\ &= \frac{\sigma^2}{\sqrt{2\pi\sigma^2 t^2}} \int_{t^2}^\infty \frac{1}{2\sigma^2} e^{-\frac{y}{2\sigma^2}} dy = \sqrt{\frac{\sigma^2}{2\pi t^2}} e^{-\frac{t^2}{2\sigma^2}}. \end{aligned}$$

On the other hand, denoting $t_\sigma = t/\sigma$ and using l'Hôpital's rule,

$$\lim_{t\to\infty} \frac{G_{0,\sigma^2}(t)}{\sqrt{\frac{\sigma^2}{2\pi t^2}} e^{-\frac{t^2}{2\sigma^2}}} = \lim_{t_\sigma\to\infty} \frac{\int_{t_\sigma}^\infty e^{-\frac{x^2}{2}} dx}{t_\sigma^{-1} e^{-\frac{t_\sigma^2}{2}}} = 1.$$

Therefore, there exists $t_0 > 0$ such that if $t > \lambda t_0$, we have

$$G_{\theta,\tau^2}(t) \ge \frac{1}{2}\sqrt{\frac{\tau^2}{2\pi t^2}} e^{-\frac{(t+\lambda(1-\gamma)^{-1})^2}{2\tau^2}} = \sqrt{\frac{\tau^2}{8\pi t^2}} e^{-\frac{(t+\lambda(1-\gamma)^{-1})^2}{2\tau^2}}.$$

By Theorem 2.3, $\mathbb{P}(X > t) = E_P\big[P_\omega(X > t)\big] = E_P\big[G_{\theta,\tau^2}(t)\big]$. To get the upper bound, observe that

$$E_P\big[G_{\theta,\tau^2}(t)\big] \le E_P\Big[\sqrt{\frac{\lambda^2}{2\pi t^2}} e^{-\frac{(t-\lambda(1-\gamma)^{-1})^2}{2\tau^2}}\Big] \le \sqrt{\frac{\lambda^2}{2\pi t^2}} E_P\big[e^{-\frac{(t-\lambda(1-\gamma)^{-1})^2}{2\tau^2}}\big].$$

Therefore,

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}(X > t) &\le \lim_{t\to\infty} \frac{1}{t^2} \log\Big(\big\|e^{-\frac{1}{2\tau^2}}\big\|_{t^2}\Big)^{t^2} = \log\Big(\big\|e^{-\frac{1}{2\tau^2}}\big\|_\infty\Big) \\ &= \log\big(e^{-\frac{1}{2\|\tau\|_\infty^2}}\big) = -\frac{1}{2\|\tau\|_\infty^2}. \end{aligned}$$

For the lower bound, we first observe that for $t > \lambda t_0$,

$$E_P\big[G_{\theta,\tau^2}(t)\big] \ge E_P\Big[\sqrt{\frac{\tau^2}{8\pi t^2}} e^{-\frac{t^2}{2\tau^2}}\Big].$$

Now, let $\varepsilon > 0$ be any positive real number such $P(\tau > \varepsilon) > 0$. Then

$$E_P\big[G_{\theta,\tau^2}(t)\big] \ge E_P\Big[\sqrt{\frac{\tau^2}{8\pi t^2}} e^{-\frac{t^2}{2\tau^2}} \cdot \mathbf{1}_{\{\tau>\varepsilon\}}\Big] \ge \sqrt{\frac{\varepsilon^2}{8\pi t^2}} E_P\Big[e^{-\frac{t^2}{2\tau^2}} \cdot \mathbf{1}_{\{\tau>\varepsilon\}}\Big],$$

which implies

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}(X > t) &\ge \lim_{t\to\infty} \frac{1}{t^2} \log\Big(\big\|e^{-\frac{1}{2\tau^2}} \cdot \mathbf{1}_{\{\tau>\varepsilon\}}\big\|_{t^2}\Big)^{t^2} = \log\Big(\big\|e^{-\frac{1}{2\tau^2}} \cdot \mathbf{1}_{\{\tau>\varepsilon\}}\big\|_\infty\Big) \\ &= \log\big(e^{-\frac{1}{2\|\tau\|_\infty^2}}\big) = -\frac{1}{2\|\tau\|_\infty^2}. \end{aligned}$$

This completes the proof of the theorem. $\square$

Next, we give a simple example of the situation when the distribution of τ can be explicitly computed, and the tails of X do not have the Gaussian asymptotic structure.

Example 2.7. *Let Assumption 1.2 hold and suppose that $P(\mu_0 = 0) = 1$, $|\mathcal{D}| = 2$, and*

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Further, assume that $\sigma_{1,n}^2$ and $\sigma_{2,n}^2$ have strictly asymmetric α-stable distributions with index $\alpha \in (0,1)$ and "Laplace transform" given by

$$E_P\big[e^{-\lambda\sigma_{i,n}^2}\big] = e^{-\theta_i\lambda^\alpha}, \qquad \lambda > 0,\ i = 1,2,$$

for some positive constants θ_i, $\theta_1 \neq \theta_2$. In notation of [30], these distributions belong to the class $S_\alpha(\theta, 1, 0)$ (see Section 1.1 and also Propositions 1.2.11 and 1.2.12 in [30]). The stationary distribution of the underlying Markov chain is uniform on $\mathcal{D}$, and therefore for the Laplace transform of the limiting variance τ^2 introduced in (7) we have for any $\lambda > 0$,

$$\begin{aligned}
\mathbb{E}\big[e^{-\lambda\tau^2}\big] &= \frac{1}{2}\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k}\sigma_{1,0}^2}\Big]\cdot\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k+2}\sigma_{2,0}^2}\Big] \\
&\quad +\frac{1}{2}\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k}\sigma_{2,0}^2}\Big]\cdot\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k+2}\sigma_{1,0}^2}\Big] \\
&= \frac{1}{2}\prod_{k=0}^{\infty}e^{-\theta_1\lambda^\alpha\gamma^{4k\alpha}}\cdot e^{-\theta_2\lambda^\alpha\gamma^{(4k+2)\alpha}} + \frac{1}{2}\prod_{k=0}^{\infty}e^{-\theta_2\lambda^\alpha\gamma^{4k\alpha}}\cdot e^{-\theta_1\lambda^\alpha\gamma^{(4k+2)\alpha}} \\
&= \frac{1}{2}e^{-\frac{\lambda^\alpha(\theta_1+\theta_2\gamma^{2\alpha})}{1-\gamma^{4\alpha}}} + \frac{1}{2}e^{-\frac{\lambda^\alpha(\theta_2+\theta_1\gamma^{2\alpha})}{1-\gamma^{4\alpha}}}.
\end{aligned}$$

Therefore, Theorem 2.3 yields for $t \in \mathbb{R}$,

$$\mathbb{E}\big[e^{itX}\big] = \mathbb{E}\Big[e^{-\frac{\tau^2t^2}{2}}\Big] = \frac{1}{2}e^{-\frac{|t|^{2\alpha}(\theta_1+\theta_2\gamma^{2\alpha})}{2^\alpha(1-\gamma^{4\alpha})}} + \frac{1}{2}e^{-\frac{|t|^{2\alpha}(\theta_2+\theta_1\gamma^{2\alpha})}{2^\alpha(1-\gamma^{4\alpha})}}.$$

Thus X is a mixture of two symmetric (2α)-stable distributions. In particular, in contrast to the result obtained under the conditions of Theorem 2.6, X has power tails. Namely (see Property 1.2.15 on p. 16 of [30]) the following limits exist, are equivalent, and are both finite and strictly positive: $\lim_{t\to\infty} t^{2\alpha}\cdot\mathbb{P}(X > t) = \lim_{t\to\infty} t^{2\alpha}\cdot\mathbb{P}(X < -t) \in (0,\infty)$.

2.3 Covariance structure of $(X_n)_{n\in\mathbb{Z}}$ in a fixed environment

This section is devoted to a characterization in a given environment of the Gaussian structure of the stationary solution $(X_n)_{n\in\mathbb{Z}}$ to (1). Using (8), we obtain for any sequence of real constants $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$:

$$\sum_{k=0}^{n} c_kX_k = \sum_{k=0}^{n} c_k\sum_{j=-\infty}^{k}\gamma^{k-j}\xi_j = \sum_{j=-\infty}^{0}\xi_j\sum_{k=0}^{n}c_k\gamma^{k-j} + \sum_{j=1}^{n}\xi_j\sum_{k=j}^{n}c_k\gamma^{k-j},$$

where we used the absolute convergence of the series to interchange the summation signs. Therefore, under the measure P_ω, that is in a given environment ω,

$$\sum_{k=0}^{n} c_k X_k \sim \mathcal{N}(\chi_{\mathbf{c},n}, \eta^2_{\mathbf{c},n}),$$

where

$$\chi^2_{\mathbf{c},n} = \sum_{j=-\infty}^{0} \mu_j \sum_{k=0}^{n} c_k \gamma^{k-j} + \sum_{j=1}^{n} \mu_j \sum_{k=j}^{n} c_k \gamma^{k-j}.$$

and

$$\eta^2_{\mathbf{c},n} = \sum_{j=-\infty}^{0} \sigma_j^2 \Big(\sum_{k=0}^{n} c_k \gamma^{k-j}\Big)^2 + \sum_{j=1}^{n} \sigma_j^2 \Big(\sum_{k=j}^{n} c_k \gamma^{k-j}\Big)^2.$$

This shows that under P_ω, the process $(X_n)_{n\geq 0}$ is Gaussian. Note that Theorem 2.2 ensures the almost sure convergence of the infinite series in the formulas above.

The following corollary is immediate from Theorem 2.3.

Corollary 2.8. *Let Assumption 1.1 hold. Then, provided that the moments in the right-hand side exist, we have the following identities:*

(i) $\mathbb{E}[X] = \dfrac{E_P[\mu_0]}{1-\gamma}$.

(ii) $\mathrm{VAR}_{\mathbb{P}}(X) = \dfrac{E_P[\sigma_0^2]}{1-\gamma^2} + \mathrm{VAR}_P(\theta)$.

Proof. It follows from Theorem 2.3 that

$$m_X := \mathbb{E}[X] = E_P\big[E_\omega\big[\mathcal{N}(\theta,\tau^2)\big]\big] = E_P[\theta] = \frac{E_P[\mu_0]}{1-\gamma}$$

and

$$\begin{aligned} \mathrm{VAR}_{\mathbb{P}}(X) &= E_P\big[E_\omega\big[X^2 - m_X^2\big]\big] = E_P[\tau^2 + \theta^2] - m_X^2 \\ &= E_P[\tau^2] + \mathrm{VAR}_P(\theta) = \frac{E_P[\sigma_0^2]}{1-\gamma^2} + \mathrm{VAR}_P(\theta), \end{aligned}$$

where we used the fact $m_X = E_P[\theta]$, and therefore $\mathrm{VAR}_P(\theta) = E[\theta^2] - m_X^2$. □

In the case of Markovian environment, $\mathrm{VAR}_P(\theta)$ can be expressed in terms of certain explicit transformations of the transition kernel of the underlying Markov chain. In the following lemma we compute $\mathrm{VAR}_P(\theta)$ under Assumption 1.2. To state the result we first need to introduce some notation. Denote by H transition matrix of the underlying Markov chain, that is

$$H(i,j) = P(y_1 = j | y_0 = i), \qquad i,j \in \mathcal{D}.$$

Denote by $\pi = \big(\pi_1, \ldots, \pi_d\big)$ the stationary distribution of $(y_n)_{n\in\mathbb{Z}}$. Let $\bar{\mu}_i = E_P[\mu_{i,0}]$ and $a = \sum_{i=1}^d \pi_i \bar{\mu}_i = E_P[\mu_0]$. Let $\mathfrak{m}_2$ denote the d-dimensional vector whose i-th component is $\pi_i \bar{\mu}_i^2$ and introduce a $d \times d$ matrix K_γ by setting

$$K_\gamma(i,j) = \frac{\gamma}{\bar{\mu}_i} \cdot H(i,j) \cdot \bar{\mu}_j, \qquad i,j = 1,\ldots,d.$$

We have:

Lemma 2.9. *Let Assumption 1.2 hold. Then*

$$\mathrm{VAR}_P(\theta) = \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2\gamma}{1-\gamma^2} \cdot \langle \mathfrak{m}_2, (I-K_\gamma)^{-1}\mathbf{1}\rangle - \frac{2\gamma a^2}{(1-\gamma^2)(1-\gamma)},$$

where $\langle \mathbf{x}, \mathbf{y}\rangle$ stands for the usual scalar product of two d-vectors $\mathbf{x}$ and $\mathbf{y}$.

Proof. For $n \in \mathbb{Z}$, let $\nu_n = \mu_{-n} - a$ and

$$\rho_n := E_P[\nu_i \nu_{n+i}] = \mathrm{COV}_P(\mu_{-i}, \mu_{-i-n}).$$

Then, according to (7),

$$\begin{aligned} \mathrm{VAR}_P(\theta) &= E_P\Big[\Big(\sum_{n=0}^\infty \gamma^n \nu_n\Big)^2\Big] = E_P\Big[\sum_{n=0}^\infty \gamma^{2n}\nu_n^2\Big] + 2\sum_{n=0}^\infty \gamma^n \sum_{k=n+1}^\infty \gamma^k \rho_{k-n} \\ &= \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + 2\sum_{n=0}^\infty \gamma^n \sum_{m=1}^\infty \gamma^{n+m}\rho_m = \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2}{1-\gamma^2} \cdot \sum_{m=1}^\infty \gamma^m \rho_m. \end{aligned}$$

It remains to compute ρ_n for $n \geq 1$. We have

$$\begin{aligned} \rho_n &= \sum_{i=1}^d \sum_{j=1}^d \pi_i H^{n-1}(i,j) E\Big[(\bar{\mu}_i - a)(\bar{\mu}_j - a)\Big] \\ &= \sum_{i=1}^d \sum_{j=1}^d \pi_i H^{n-1}(i,j) E_P\Big[(\bar{\mu}_i\bar{\mu}_j - a\bar{\mu}_i - a\bar{\mu}_j + a^2)\Big] \\ &= \sum_{i=1}^d \sum_{j=1}^d \pi_i H^{n-1}(i,j) E_P\big[\bar{\mu}_i\bar{\mu}_j - a^2\big] = E\Big[\sum_{i=1}^d \sum_{j=1}^d \pi_i \bar{\mu}_i H^{n-1}(i,j)\bar{\mu}_j\Big] - a^2. \end{aligned}$$

Define the following *Doob transform* of matrix H :

$$K(i,j) = \frac{1}{\bar{\mu}_i} H(i,j)\bar{\mu}_j, \qquad i,j = 1,\ldots,d.$$

Then, a routine induction argument shows that for any $n \in \mathbb{N}$, $K^n(i,j) = \frac{1}{\bar{\mu}_i} H^n(i,j)\bar{\mu}_j$. Using this formula, we obtain

$$\rho_n = E_P\Big[\sum_{i=1}^d \sum_{j=1}^d \pi_i \bar{\mu}_i^2 K^{n-1}(i,j)\Big] - a^2 = \langle \mathfrak{m}_2, K^{n-1}\mathbf{1}\rangle - a^2,$$

and hence

$$\begin{aligned}\mathrm{VAR}_P(\theta) &= \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2}{1-\gamma^2}\cdot\sum_{n=1}^{\infty}\gamma^n\big(\langle\mathfrak{m}_2, K^{n-1}\mathbf{1}\rangle - a^2\big)\\ &= \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2\gamma}{1-\gamma^2}\cdot\langle\mathfrak{m}_2,(I-K_\gamma)^{-1}\mathbf{1}\rangle - \frac{2\gamma a^2}{(1-\gamma^2)(1-\gamma)},\end{aligned}$$

The proof of the lemma is completed. □

Remark 2.10. *It is not hard to verify that with an appropriate modification of the definition of the Doob transform K_γ (as a positive integral kernel rather than a d-matrix), the statement of Lemma 2.9 remains true for a general, non-necessarily restricted to a finite-state, Markovian setup.*

In the reminder of this section we assume, for simplicity, that $P(\mu_n = 0) = 1$. The distribution of a mean-zero Gaussian sequence is entirely determined by its covariance structure. It follows from (5) that

$$X_{k+n} = \gamma^n X_k + \sum_{t=0}^{n-1}\gamma^t\xi_{n+k-t-1}, \qquad k\in\mathbb{Z},\ n\in\mathbb{N}.$$

Therefore, for any $k\in\mathbb{Z}$ and $n\in\mathbb{N}$, we have

$$\mathrm{COV}_\omega(X_kX_{k+n}) = E_\omega[X_kX_{k+n}] = \gamma^n E_\omega[X_k^2]. \tag{9}$$

In particular, random variables X_n and X_m are positively correlated for any $n,m\in\mathbb{Z}$.

3 Asymptotic behavior of X when $\gamma\to 1^-$.

To emphasize the dependence of the stationary solution to (1) on γ, we throughout this section use the notation X_γ for X. To illustrate the main result of this section, consider first the case when the coefficients ξ_n in (1) are independent and distributed according to $\mathcal{N}(0,\sigma^2)$ for some constant $\sigma>0$. Then $X_\gamma \sim \frac{1}{\sqrt{1-\gamma^2}}\mathcal{N}(0,\sigma^2)$, and hence,

$$\sqrt{1-\gamma}\cdot X_\gamma \overset{\mathbb{P}}{\Longrightarrow} \frac{1}{\sqrt{2}}\mathcal{N}(0,\sigma^2), \qquad \text{as } \gamma\to 1^-. \tag{10}$$

We next show that in a certain sense $(1-\gamma)^{-1/2}$ is always the proper scaling factor for the distribution of X_γ when $\gamma\to 1^-$.

Theorem 3.1. *Let Assumption 1.1 hold. Then* $\dfrac{\log|X_\gamma|}{\log(1-\gamma)} \xrightarrow{\mathbb{P}} -\dfrac{1}{2}$ *as* $\gamma\to 1^-$, *where* $\xrightarrow{\mathbb{P}}$ *means convergence in probability under the law* $\mathbb{P}$.

Proof. We must prove that for any $\varepsilon>0$,

$$\mathbb{P}\Big(\Big|\frac{\log|X_\gamma|}{\log(1-\gamma)}+\frac{1}{2}\Big|>\varepsilon\Big) \to_{\gamma\to 1^-} 0 \tag{11}$$

This is equivalent to the following two claims:

$$\mathbb{P}\Big(\frac{\log|X_\gamma|}{\log(1-\gamma)} > -\frac{1}{2}+\varepsilon\Big) \to_{\gamma\to 1^-} 0 \quad \text{and} \quad \mathbb{P}\Big(\frac{\log|X_\gamma|}{\log(1-\gamma)} < -\frac{1}{2}-\varepsilon\Big) \to_{\gamma\to 1^-} 0.$$

Since $\log(1-\gamma)<0$, it suffices to show that

$$\mathbb{P}\Big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}\Big) \to_{\gamma\to 1^-} 0 \quad \text{and} \quad \mathbb{P}\Big(|X_\gamma| < (1-\gamma)^{-\frac{1}{2}+\varepsilon}\Big) \to_{\gamma\to 1^-} 0.$$

Toward this end, observe that due to Assumption 1.1 the distribution of τ does not have an atom at zero, i.e. $P(\tau>0)=1$. Therefore,

$$\lim_{\delta\to 0} P\big(\tau^2 \notin (\delta,\delta^{-1})\big) = 0, \tag{12}$$

Fix now arbitrary $\varepsilon>0$ and $\delta\in(0,1)$, and let $I_\delta := (\delta,\delta^{-1})$. We have

$$\begin{aligned}
&\mathbb{P}\big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}\big)\\
&\quad = \mathbb{P}\big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}; \tau\in I_\delta\big) + \mathbb{P}\big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}; \tau\notin I_\delta\big)\\
&\quad \le \mathbb{P}\big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}; \tau^2\in I_\delta\big) + P\big(\tau^2\notin I_\delta\big).
\end{aligned}$$

Similarly,

$$\mathbb{P}\Big(|X_\gamma| < (1-\gamma)^{-\frac{1}{2}+\varepsilon}\Big) \to_{\gamma\to 1^-} \le \mathbb{P}\big(|X_\gamma| < (1-\gamma)^{-\frac{1}{2}+\varepsilon}; \tau^2\in I_\delta\big) + P\big(\tau^2\notin I_\delta\big)$$

Using the "δ-truncated version" of τ^2, we obtain

$$\begin{aligned}
\mathbb{P}\big(|X_\gamma|\sqrt{1-\gamma} < (1-\gamma)^{\varepsilon}; \tau^2\in I_\delta\big) &= E_P\big[P_\omega\big(|X_\gamma|\sqrt{1-\gamma} < (1-\gamma)^{\varepsilon}\big); \tau^2\in I_\delta\big]\\
&\ge P\big(|\mathcal{N}(0,\delta)| < (1-\gamma)^{\varepsilon}\big) \to_{\gamma\to 1^-} 0,
\end{aligned} \tag{13}$$

and, similarly,

$$\mathbb{P}\big(|X_\gamma|\sqrt{1-\gamma} > (1-\gamma)^{-\varepsilon}; \tau^2\in I_\delta\big) \le P\big(|\mathcal{N}(0,\delta^{-1})| < (1-\gamma)^{\varepsilon}\big) \to_{\gamma\to 1^-} 0. \tag{14}$$

Taking in account that $\delta\in(0,1)$ is arbitrary, and combining (13) and (14) together with (12), we obtain (11). The proof of the theorem is completed. $\square$

4 Extreme values of $(X_n)_{n\ge 0}$

The goal of this section is twofold. First, we prove a limit theorem for the running maxima $M_n = \max_{1\le k\le n} X_k$ (Theorem 4.1 below). This result provides some information about the first passage times $T_a = \inf\{t>0 : X_t > a\}$, through the identity of the events $\{T_a > n\}$ and $\{\max_{k\le n} X_k < a\}$. Next, we obtain a law of iterated logarithm type result for the sequence $(X_n)_{n\in\mathbb{N}}$ (Theorem 4.3 below).

There is an extensive literature discussing asymptotic behavior of maxima of Gaussian processes. The following general result suffices for our purposes (see [34] or Theorem A in [7]). For $n\in\mathbb{N}$, let

$$a_n = \sqrt{2\log n} \qquad \text{and} \qquad b_n = a_n - \frac{\log a_n + \log\sqrt{2\pi}}{a_n}. \tag{15}$$

Theorem 4.1. *[34] Let $(X_n)_{n\in\mathbb{Z}}$ be a Gaussian sequence with $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^2] = 1$. Let $\rho_{ij} = \mathbb{E}[X_iX_j]$ and $M_n = \max_{1\le k\le n} X_k$. If*

(i) $\delta := \sup_{i<j} |\rho_{ij}| < 1$.

(ii) For some $\lambda > \frac{2(1+\delta)}{1-\delta}$,

$$\frac{1}{n^2}\sum_{1\le i<j\le n} |\rho_{ij}|\cdot\log(j-i)\cdot\exp\big\{\lambda|\rho_{ij}|\cdot\log(j-i)\big\}\to 0, \qquad \text{as } n\to\infty, \tag{16}$$

then, for any $y\in\mathbb{R}$, $\mathbb{P}\big(M_n \le b_n + a_n^{-1}y\big)\to\exp\big\{-e^{-y}\big\}$ as $n\to\infty$.

The theorem implies a sharp concentration of the running maximum around its long-term asymptotic average a_n. The limiting distribution in Theorem 4.1 is called the *standard Gumbel distribution* (cf. [20]). We next study the asymptotic distribution of the random variables

$$L_n = \max_{0\le k\le n}\frac{X_k}{\lambda_k} \qquad \text{and} \qquad M_n = \max_{0\le k\le n} X_k, n\in\mathbb{N}$$

under Assumption 1.1. Let

$$\lambda_k^2 := E_\omega[X_k^2] = \sum_{j=0}^{\infty}\gamma^{2j}\sigma_{k-j}^2, \qquad k\in\mathbb{Z}. \tag{17}$$

We have:

Theorem 4.2. *Let Assumption 1.1 hold. Suppose in addition that $E_P[\mu_0] = 0$ and*

$$P\big(\sigma_0\in(\delta,\delta^{-1})\big) = 1 \tag{18}$$

for some constant $\delta\in(0,1)$. Then

(a) For any constant $y\in\mathbb{R}$,

$$\lim_{n\to\infty} P_\omega\big(a_n(L_n-b_n)\le y\big) = \exp\{-e^{-y}\}, \qquad P-\text{a.s.}, \tag{19}$$

where a_n and b_n are defined in (15).

(b) Further,

$$\frac{\log M_n}{\log\log n}\xrightarrow{P_\omega}\frac{1}{2}, \qquad P-\text{a.s.}$$

Proof.
(a) Let $U_k = \frac{X_k}{\lambda_k}$, $k\in\mathbb{Z}$. Then $E_\omega[U_k] = 0$ and $E_\omega[U_k^2] = 1$. Furthermore, (9) implies for any $k\in\mathbb{Z}$ and $n\in\mathbb{N}$,

$$\rho_{n,k+n} := \mathrm{COV}_\omega(U_kU_{k+n}) = E_\omega[U_kU_{k+n}] = \gamma^n\frac{\lambda_k}{\lambda_{k+n}}. \tag{20}$$

It suffices to verify that the conditions of Theorem 4.1 are satisfied for random variables U_n. Toward this end, observe that (17) implies

$$\lambda_{k+n}^2 = \gamma^{2n}\lambda_k^2 + \sum_{t=0}^{n-1} \gamma^{2t}\sigma_{k+n-t-1}^2,$$

and hence, in virtue of (17) and (18),

$$\frac{\lambda_{k+n}}{\gamma^n \lambda_k} = \sqrt{1+\gamma^{-2n}\lambda_k^{-2}\sum_{t=0}^{n-1}\gamma^{2t}\sigma_{k+n-t-1}^2} > \sqrt{1+\gamma^{-2n}\lambda_k^{-2}\gamma^{2n-2}\sigma_k^2} > \sqrt{1+\gamma^{-2}\delta^4}.$$

Thus

$$\mathfrak{r} := \sup_{k\in\mathbb{Z},n\in\mathbb{N}} \rho_{k,k+n} = \sup_{k\in\mathbb{Z},n\in\mathbb{N}} \Big\{\gamma^n \frac{\lambda_k}{\lambda_{k+n}}\Big\} < 1.$$

Furthermore, it follows from (20) and (17) that, under condition (18), we have for any constant $\mathfrak{s} > 0$:

$$\begin{aligned}
&\frac{1}{n^2}\sum_{1\le i<j\le n} |\rho_{ij}|\cdot\log(j-i)\cdot\exp\big\{\mathfrak{s}|\rho_{ij}|\cdot\log(j-i)\big\}\\
&\quad\le \frac{1}{n^2}\sum_{1\le i<j\le n}\frac{1}{\delta^4}\gamma^{(j-i)}\cdot\log(j-i)\cdot\exp\big\{\mathfrak{s}\delta^{-4}\cdot\log(j-i)\big\}\\
&\quad= \frac{1}{n^2\delta^4}\sum_{1\le i<j\le n}\gamma^{(j-i)}\log(j-i)\cdot(j-i)^{\mathfrak{s}\delta^{-4}} = \frac{1}{n^2\delta^4}\sum_{k=1}^{n-1}(n-k)\cdot\gamma^k\log k\cdot k^{\mathfrak{s}\delta^{-4}}\\
&\quad\le \frac{1}{n\delta^4}\sum_{k=1}^{\infty}\gamma^k\log k\cdot k^{\mathfrak{s}\delta^{-4}} \to 0, \qquad \text{as } n\to\infty.
\end{aligned}$$

Therefore, (19) holds for any $y\in\mathbb{R}$ by Theorem 4.1. The proof of part (a) of the theorem is completed.

(b) It follows from the conditions of the theorem that there exists $c_0 > 0$ such that for all $n\in\mathbb{Z}$,

$$c_0^{-1} < \frac{M_n}{L_n} < c_0, \qquad P-\text{a.s.}$$

Therefore, $P-$a.s., for any $\varepsilon > 0$, we have

$$P_\omega\Big(\frac{\log M_n}{\log\log n} > \frac{1}{2}+\varepsilon\Big) \;=\; P_\omega\big(M_n > (\log n)^{\frac{1}{2}+\varepsilon}\big) \le P_\omega\big(L_n > c_0(\log n)^{\frac{1}{2}+\varepsilon}\big).$$

Part (a) of the theorem implies that, for any $y\in\mathbb{R}$,

$$\lim_{n\to\infty} P_\omega\big(L_n \le y a_n^{-1} + b_n\big) = \exp\{-e^{-y}\} \qquad P-\text{a.s.} \tag{21}$$

Since for any fixed $y > 0$ and $\varepsilon > 0$, eventually (for all n, large enough) we have

$$ya_n^{-1} + b_n < c_0(\log n)^{\frac{1}{2}+\varepsilon}.$$

It follows from (21) (because we can use arbitrarily small y while $\lim_{y\to-\infty} \exp\{-e^{-y}\} = 0$) that

$$\lim_{n\to\infty} P_\omega\Big(\frac{\log M_n}{\log\log n} > \frac{1}{2} + \varepsilon\Big) = 0 \qquad P-\text{a.s.}$$

Similarly, since $P-$ a.s., for any $\varepsilon > 0$,

$$P_\omega\Big(\frac{\log M_n}{\log\log n} < 1 - \varepsilon\Big) \quad = \quad P_\omega\big(M_n < (\log n)^{\frac{1}{2}-\varepsilon}\big) \le P_\omega\big(L_n < c_0^{-1} \cdot (\log n)^{\frac{1}{2}-\varepsilon}\big),$$

while for any $y \in \mathbb{R}$, eventually,

$$c_0^{-1} \cdot (\log n)^{1/2-\varepsilon} < ya_n^{-1} + b_n,$$

It follows from (21), using this time arbitrarily large values of y, that

$$\lim_{n\to\infty} P_\omega\Big(\frac{\log M_n}{\log\log n} < \frac{1}{2} - \varepsilon\Big) = 0 \qquad P-\text{a.s.}$$

The proof of the theorem is completed. □

We next prove a "law of iterated logarithm"-type asymptotic result for the sequence X_n. We have:

Theorem 4.3. *Let the conditions of Theorem 4.2 hold. Let $(X_n)_{n\ge1}$ be the stationary solution to* (1) *defined by* (6)*. Then there exists a constant $c > 0$ such that*

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = c, \qquad \mathbb{P}-\text{a.s.}$$

Proof. The claim follows from the bounds provided by a coupling of X_n with the following "extremal versions" of it. Let $(U_n)_{n\in\mathbb{Z}}$ and $(V_n)_{n\in\mathbb{Z}}$ be two stationary sequences that satisfy, respectively,

$$U_{n+1} = \gamma U_n + \delta^{-1}\varepsilon_n \qquad \text{and} \qquad V_{n+1} = \gamma V_n + \delta\varepsilon_n,$$

where δ is the constant introduced in the conditions of Theorem 4.2. Notice that, for all $n \in \mathbb{Z}$, we have $\text{VAR}_\mathbb{P}(U_n) = \delta^2 \cdot (1-\gamma^2)^{-1}$ and $\text{VAR}_\mathbb{P}(V_n) = \delta^{-2} \cdot (1-\gamma^2)^{-1/2}$. Furthermore, it follows for instance from (9) that

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \text{COV}_\mathbb{P}(U_k, U_{n+k}) = \lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \text{COV}_\mathbb{P}(V_k, V_{n+k}) = 0$$

Therefore, Theorem 2 in [21] implies that, with probability one,

$$\limsup_{n\to\infty} \frac{V_n}{\sqrt{2\log n}} = \delta \cdot (1-\gamma^2)^{-1/2} \quad \text{and} \quad \limsup_{n\to\infty} \frac{U_n}{\sqrt{2\log n}} = \delta^{-1} \cdot (1-\gamma^2)^{-1/2}.$$

For an event A, let A^c denote the complement of A. Since the event $\{\limsup_{n\to\infty} W_n = a\}$ for a sequence of random variables W_n can be represented as the intersection of the following two events:

$$\bigcap_{\varepsilon>0} \{W_n > a - \varepsilon \text{ i. o.}\} \qquad \text{and} \qquad \bigcap_{\varepsilon>0} \{W_n > a + \varepsilon \text{ i. o.}\}^c,$$

then an application of the conditional Borel-Cantelli lemma (see, for instance, [13, p. 240]) with the filtration $\mathcal{F}_n = \sigma(\varepsilon_k, \sigma_k : k \le n)$ yields that the following inequalities hold with probability one:

$$\limsup_{n\to\infty} \frac{V_n}{\sqrt{2\log n}} \le \limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} \le \limsup_{n\to\infty} \frac{U_n}{\sqrt{2\log n}}.$$

Thus, there exists a function $c(X)$ of $X = (X_n)_{n\in\mathbb{N}}$ such that, with probability one,

$$\delta \cdot (1-\gamma^2)^{-1/2} \le \limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = c(X) \le \delta^{-1} \cdot (1-\gamma^2)^{-1/2}.$$

The fact that $c(X)$ is actually a constant function follows from the ergodicity of the sequence X_n (which is implied by (6) along with the ergodicity of the sequence ξ_n) and the shift invariance of the limiting constant. By the latter we mean that

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = \limsup_{n\to\infty} \frac{X_{n+1}}{\sqrt{2\log n}}.$$

The proof of the theorem is completed. □

5 Random walk $S_n = \sum_{k=1}^n X_k$

This section includes limit theorems describing the asymptotic properties of $S_n = \sum_{k=1}^n X_k$. Specifically, we prove a law of large numbers (Theorem 5.2), associated with it large deviation bounds (Theorem 5.3 and , Corollary 5.4,), and central limit theorems (Theorem 5.5 and Theorem 5.6) for the sequence S_n.

Random walk $S_n = \sum_{k=1}^n X_k$ associated with Equation (1) has been studied in [27] and [28]. The following decomposition of S_n, which is implied by (5), is useful:

$$\begin{aligned} S_n &= \sum_{k=1}^n \gamma^k X_0 + \sum_{k=1}^n \sum_{t=1}^k \gamma^{k-t}\xi_t = \sum_{k=1}^n \gamma^k X_0 + \sum_{t=1}^n \sum_{k=t}^n \gamma^{k-t}\xi_t \\ &= \sum_{k=1}^n \gamma^k X_0 + \sum_{t=1}^n \Big(\sum_{k=t}^\infty \gamma^{k-t} - \sum_{k=n+1}^\infty \gamma^{k-t} \Big) \xi_t \\ &= \sum_{k=1}^n \gamma^k X_0 + (1-\gamma)^{-1} \sum_{t=1}^n \xi_t - (1-\gamma)^{-1} \sum_{t=1}^n \gamma^{n+1-t}\xi_t. \end{aligned} \tag{22}$$

Similar decompositions have been used, for instance, in [28] and [27]. Due to Assumption 1.1, the following inequalities hold with probability one (the right-most inequality in (24) is implied by Theorem 2.2):

$$\Big|\sum_{k=1}^{n}\gamma^k X_0\Big| \;\le\; |X_0|\cdot\sum_{k=0}^{\infty}\gamma^k<\infty, \tag{23}$$

and

$$\Big|\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t\Big| \stackrel{D}{=} \Big|\sum_{t=-n+1}^{0}\gamma^{1-t}\xi_t\Big| \le \sum_{k=0}^{\infty}\gamma^{k+1}\cdot|\xi_{-k}|<\infty, \tag{24}$$

where $\stackrel{D}{=}$ means equivalence of distributions. This shows that only the second term in the right-most expression of (22) contributes to the asymptotic behavior of S_n. More precisely, we have the following lemma. Though the proof of the lemma is by standard arguments, we provide it below for the reader's convenience.

Lemma 5.1. *Let Assumption 1.1 hold. Then*

(a) For any sequence of reals $(a_n)_{n\in\mathbb{N}}$ increasing to infinity, we have

$$\frac{1}{a_n}\sum_{k=1}^{n}\gamma^k X_0 \to_{n\to\infty} 0, \qquad \mathbb{P}-\text{a.s.}$$

and

$$\frac{1}{a_n}\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t \to_{n\to\infty} 0, \qquad \text{in probability.}$$

(b) If in addition $E_P\big[|\mu_0|\big]<\infty$ and $E_P\big[\sigma_0\big]<\infty$, then

$$\frac{1}{n}\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t \to_{n\to\infty} 0, \qquad \mathbb{P}-\text{a.s.}$$

Proof.
(a) The first claim of part (a) is a direct consequence of (23). The second claim follows from (24) as follows. For any $\varepsilon>0$, we have in virtue of (24),

$$\mathbb{P}\Big(\frac{1}{a_n}\Big|\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t\Big|>\varepsilon\Big)=$$
$$=\mathbb{P}\Big(\frac{1}{a_n}\Big|\sum_{t=-n+1}^{0}\gamma^{1-t}\xi_t\Big|>\varepsilon\Big)\le\mathbb{P}\Big(\frac{1}{a_n}\sum_{k=0}^{\infty}\gamma^{k+1}\cdot|\xi_{-k}|>\varepsilon\Big)\to_{n\to\infty}0,$$

which implies the result.

(b) We must show that for any $\varepsilon>0$,

$$\mathbb{P}\Big(\frac{1}{n}\Big|\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t\Big|>\varepsilon \text{ i.o.}\Big)=0,$$

where the abbreviation "i.o." stands for *infinitely often*. By the Borel-Cantelli lemma, it suffices to show that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\frac{1}{n}\Big|\sum_{t=1}^{n} \gamma^{n+1-t}\xi_t\Big| > \varepsilon\Big) < \infty. \tag{25}$$

Using (24), we obtain

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\frac{1}{n}\Big|\sum_{t=1}^{n} \gamma^{n+1-t}\xi_t\Big| > \varepsilon\Big) \le \mathbb{P}\Big(\frac{1}{n}\sum_{k=0}^{\infty} \gamma^{k+1}\cdot|\xi_{-k}| > \varepsilon\Big) \le \frac{1}{\varepsilon}\mathbb{E}\Big[\sum_{k=0}^{\infty}\gamma^{k+1}\cdot|\xi_{-k}|\Big]$$
$$= \frac{\gamma}{\varepsilon(1-\gamma)}\mathbb{E}\big[|\xi_0|\big].$$

Since ξ_k are Gaussian random variables under P_ω, implies

$$\mathbb{E}\big[|\xi_0|\big] = E_P\Big[|\mu_0| + \sqrt{\frac{2\sigma_0^2}{\pi}}\Big]. \tag{26}$$

It hence follows from the conditions of the lemma that $\mathbb{E}\big[|\xi_k|\big] < \infty$. This establishes (25) and therefore completes the proof of part (b) of the lemma. □

In particular, one can obtain the following strong law of large numbers.

Theorem 5.2. *Let Assumption 1.1 hold and suppose in addition that* $E_P\big[|\mu_0|\big] < \infty$ *and* $E_P\big[\sigma_0\big] < \infty$. *Then,*

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X] = (1-\gamma)^{-1}\, E_P[\mu_0], \qquad \mathbb{P}-\text{a.s.} \tag{27}$$

Proof. Recall that under Assumption 1.1, $(\xi_n)_{n\in\mathbb{Z}}$ is stationary and ergodic sequence. Furthermore, (26) implies that $\mathbb{E}\big[|\xi_0|\big] < \infty$. Therefore, by the Birkhoff ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \xi_k = \mathbb{E}[\xi_0] = E_P[\mu_0], \qquad \mathbb{P}-\text{a.s.} \tag{28}$$

It follows now from (22) and Lemma 5.1 that

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{1}{1-\gamma}\frac{1}{n}\sum_{k=1}^{n}\xi_k = \frac{1}{1-\gamma}\, E_P[\mu_0], \qquad \mathbb{P}-\text{a.s.}$$

The proof of the theorem is completed. □

The above law of large numbers can be complemented by the following large deviation result. Recall that a sequence R_n of random variables is said to satisfy the large deviation principle (LDP) with a a lower semi-continuous *rate function* $I : \mathbb{R} \to [0,\infty]$, if for any Borel set $E \subset \mathbb{R}$,

$$-\inf_{x\in E^\circ} I(x) \le \liminf_{n\to\infty} \frac{1}{n}\log \mathbb{P}(R_n \in E) \le \limsup_{n\to\infty} \frac{1}{n}\mathbb{P}(R_n \in E) \le -\inf_{x\in \overline{E}} I(x)$$

where $\overline{E}$ and E° denote, respectively, the closure and interior of E. The rate function is *good* if the level sets $\{x \in \mathbb{R} : I(x) \le c\}$ are compact for any $c \ge 0$.

We have:

Theorem 5.3. *Let the conditions of Theorem 4.2 hold. Assume in addition that the sequence $R_n := \frac{1}{n}\sum_{k=1}^n \sigma_k^2$ satisfies the LDP with a good rate function $I(x)$. Then $\frac{S_n}{n}$ satisfies the LDP with a good rate function such that $J(x) \in (0,\infty)$ for $x \neq 0$.*

Proof. Recall $G_{\mu,\sigma^2}(t)$ from (4). The l'Hôpital rule implies that

$$\lim_{t\to\infty} \frac{G_{0,\sigma^2}(t)}{\sqrt{\frac{\sigma^2}{2\pi t^2}}e^{-\frac{t^2}{2\sigma^2}}} = \lim_{t_\sigma\to\infty} \frac{\int_{t_\sigma}^\infty e^{-\frac{x^2}{2}}dx}{t_\sigma^{-1}e^{-\frac{t_\sigma^2}{2}}} = 1.$$

Therefore, there exists $t_0 > 0$ such that $t > t_0$ implies

$$\frac{1}{2}\sqrt{\frac{\sigma^2}{2\pi t^2}}e^{-\frac{t^2}{2\sigma^2}} \le G_{0,\sigma^2}(t) \le 2\sqrt{\frac{\sigma^2}{2\pi t^2}}e^{-\frac{t^2}{2\sigma^2}}.$$

It follows from (22) and (6) that $\mathbb{P}(S_n > nt) = E_P\big[P_\omega(S_n > nt)\big] = E_P\big[G_{0,\tau_n^2}(nt)\big]$, where

$$\tau_n^2 = \frac{\gamma^2(1-\gamma^n)^2\sum_{t=0}^\infty \sigma_{-t}^2\gamma^{2t}}{(1-\gamma)^2} + \frac{\sum_{t=1}^n \sigma_t^2\big(1-\gamma^{n+1-t}\big)^2}{(1-\gamma)^2}.$$

It the follows from (18) that for any $t > 0$,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(S_n > nt) = \lim_{n\to\infty}\frac{1}{n}\log E_P\big(e^{-\frac{t^2n^2}{2\tau_n^2}}\big),$$

provided that the latter limit exists. We will next estimate the difference

$$\frac{1}{\tau_n^2} - \frac{(1-\gamma)^2}{\sum_{t=1}^n\sigma_t^2},$$

Using (18), for the lower bound we have

$$\begin{aligned}
\Big|\frac{1}{\tau_n^2} - \frac{(1-\gamma)^2}{\sum_{t=1}^n\sigma_t^2}\Big| &\le \frac{\sum_{t=1}^n\sigma_t^2 - \sum_{t=1}^n\sigma_t^2\big(1-\gamma^{n+1-t}\big)^2 + \gamma^2(1-\gamma^n)^2\sum_{t=0}^\infty\sigma_{-t}^2\gamma^{2t}}{\Big(\sum_{k=1}^n\sigma_t^2(1-\gamma^k)^2\Big)^2(1-\gamma)^{-2}} \\
&\le \frac{\delta^{-2}\Big(n - \sum_{k=1}^n(1-\gamma^k)^2 + \gamma^2(1-\gamma^2)^{-1}\Big)}{\Big(\sum_{t=1}^n\sigma_t^2\big(1-\gamma^{n+1-t}\big)^2\Big)^2(1-\gamma)^{-2}} \\
&\le \frac{\delta^{-4}\Big(2\sum_{k=1}^n\gamma^k + \gamma^2(1-\gamma^2)^{-1}\Big)}{\Big(\sum_{k=1}^n(1-\gamma^k)^2\Big)^2(1-\gamma)^{-2}} \le \frac{\delta^{-4}\Big(2\sum_{k=1}^n\gamma^k + \gamma^2(1-\gamma^2)^{-1}\Big)}{\Big(\sum_{k=1}^n(1-\gamma)^2\Big)^2(1-\gamma)^{-2}} \\
&\le n^{-2}\cdot\frac{\delta^{-4}\Big(2\sum_{k=1}^\infty\gamma^k + \gamma^2(1-\gamma^2)^{-1}\Big)}{(1-\gamma)^2}.
\end{aligned}$$

Thus

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(S_n > nt) = \lim_{n\to\infty}\frac{1}{n}\log E_P\Big(e^{-\frac{t^2n^2(1-\gamma)^2}{2\sum_{t=1}^n\sigma_t^2}}\Big),$$

provided that the limit in the right-hand side exists. In virtue of(18), we have

$$-\frac{t^2(1-\gamma)^2\delta^{-2}}{2} \leq \frac{1}{n}\log E_P\Big(e^{-\frac{t^2n^2(1-\gamma)^2}{2\sum_{t=1}^n \sigma_t^2}}\Big) \leq -\frac{t^2(1-\gamma)^2\delta^2}{2}.$$

Thus one can apply Varadhan's integral lemma (see [12, p. 137]) to $R_n := \frac{1}{n}\sum_{t=1}^n \sigma_t^2$ and continuous function $\phi_t(x) = -\frac{t^2(1-\gamma)^2}{2x} : (0,\infty) \to \mathbb{R}$. It follows from Varadhan's lemma that

$$\lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}(S_n > nt) = \lim_{n\to\infty}\frac{1}{n}\log E_P\big[e^{n\phi_t(R_n)}\big] = \sup_{x>0}\{\phi_t(x) - I(x)\}.$$

Furthermore, a symmetry argument shows that

$$\lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}(S_n > nt) = \lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}(S_n < -nt), \qquad t > 0.$$

Since $J(t) = -\sup_{x>0}\{\phi_t(x) - I(x)\} \in [0,\infty)$ is a strictly increasing function for $t \geq 0$, this implies that the LDP for S_n/n holds with rate function J (cf. [12, p. 31]).

It remains to show that J is a good rate function. Toward this end fix $c > 0$ and consider $\Psi(c) = \{t > 0 : J(t) > c\}$. Then $t \in \Psi(c)$ if and only if $t > 0$ and

$$\inf_{x>0}\Big\{\frac{t^2(1-\gamma)^2}{2x} + I(x)\Big\} > c.$$

It thus suffices to verify that $t_0 := \inf \Psi(c) \notin \Psi(c)$. Assume the contrary, that is suppose that for some $c_0 > c$

$$\inf_{x>0}\Big\{\frac{t_0^2(1-\gamma)^2}{2x} + I(x)\Big\} = c_0 > c. \tag{29}$$

Let $x_0 = \frac{t_0^2(1-\gamma)^2}{4c_0}$. We then can choose $t_1 < t_0$ such that

$$\inf_{x<x_0}\Big\{\frac{t_1^2(1-\gamma)^2}{2x} + I(x)\Big\} \geq \inf_{x<x_0}\Big\{\frac{t_1^2(1-\gamma)^2}{2x}\Big\} > c$$

and

$$\begin{aligned}&\inf_{x\geq x_0}\Big\{\frac{t_1^2(1-\gamma)^2}{2x} + I(x)\Big\}\\ &\qquad \geq \inf_{x\geq x_0}\Big\{\frac{t_0^2(1-\gamma)^2}{2x} + I(x)\Big\} - \sup_{x\geq x_0}\Big\{\frac{t_0^2(1-\gamma)^2}{2x} - \frac{t_1^2(1-\gamma)^2}{2x}\Big\} > c.\end{aligned}$$

Clearly, this contradicts (29) and hence shows that $t_0 \notin \Psi(c)$, as desired. the proof of the theorem is completed. □

The following is implied by, for instance, Theorem 3.1.2 in [12, p. 74].

Corollary 5.4. *Let Assumption 1.2 and the conditions of Theorem 4.2 hold. Then* $\frac{S_n}{n}$ *satisfies the LDP with a good rate function.*

It follows from (22) that if $E_P[\mu_0] = 0$ and $b_n^{-1}\sum_{k=1}^n \sigma_k^2$ converges in distribution to a random variable G for a suitable sequence $b_n \nearrow \infty$, then $S_n/\sqrt{b_n}$ converges in distribution to $\mathcal{N}(0,G)$. In a generic example, σ_n are in the domain of attraction of a symmetric stable law and the sequence $(\sigma_n)_{n\in\mathbb{Z}}$ satisfies certain mixing conditions. Limit theorems for S_n of this type can be found in [27]. We also refer the reader to [27] for a law of iterated logarithm for S_n. The special (Gaussian, once the environment is fixed) structure of the sequence ξ_n which is considered in this work, leads to the following result. It is different in essence from the limit theorems obtained in [27].

Theorem 5.5. *Let Assumption 1.1 hold and assume in addition that* $E_P[\mu_0] = 0$ *and* $E_P\big[\sigma_0^2\big] < \infty$. *Then,*

$$\frac{1}{\sqrt{n}} S_n \overset{\mathbb{P}}{\Longrightarrow} \frac{1}{1-\gamma}\mathcal{N}\big(0,\Sigma\big)$$

for $\Sigma := E_P[\sigma_0^2]$.

Proof. By the Birkhoff ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \sigma_k^2 = E_P[\sigma_0^2], \qquad P-\text{a.s.}$$

Hence, letting $W_n = \sum_{k=1}^n \xi_k$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\big[e^{it\frac{W_n}{\sqrt{n}}}\big] &= \lim_{n\to\infty} E_P\Big[E_\omega\big[e^{it\frac{W_n}{\sqrt{n}}}\big]\Big] = E_P\Big[\lim_{n\to\infty} E_\omega\big[e^{it\frac{W_n}{\sqrt{n}}}\big]\Big] \\ &= E_P\Big[\lim_{n\to\infty} e^{-t^2\frac{\sum_{k=1}^n \sigma_k^2}{2n}}\Big] = e^{-\frac{t^2\Sigma}{2}}.\end{aligned}$$

Therefore, $\frac{W_n}{\sqrt{n}} \overset{\mathbb{P}}{\Longrightarrow} \mathcal{N}\big(0,\Sigma\big)$. It follows now from (22) and part (a) of Lemma 5.1 that

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1-\gamma}\frac{W_n}{\sqrt{n}} = \frac{1}{1-\gamma}\mathcal{N}\big(0,\Sigma\big),$$

where the limits in the above identities are understood in terms of the convergence in distribution. The proof of the theorem is completed. □

The above theorem can be strengthened to a functional central limit result in the Skorokhod space $D[0,1]$ of *càdlàg* functions for the sequence of processes

$$J_n(t) = \frac{S_{[nt]}}{\sqrt{n\Sigma^2}}, \qquad t \in [0,1],$$

where $\Sigma^2 = \dfrac{E_P[\sigma_0^2]}{(1-\gamma)^2}$ as in the statement of Theorem 5.5, and $[x]$ denotes the integer part of $x \in \mathbb{R}$, that is $[x] = \max\{k \in \mathbb{Z} : k \le x\}$. We have:

Theorem 5.6. *Let the conditions of Theorem 4.2 hold. Then, for* P*-almost every environment* ω, *the sequence* J_n *converges in* $D[0,1]$ *under* P_ω *weakly to a standard Brownian motion. Consequently,* J_n *converges in* $D[0,1]$ *weakly to a standard Brownian motion also under* $\mathbb{P}$.

Proof. It is not hard to verify the convergence under P_ω of the finite-dimensional distributions of J_n to those of a standard Brownian notion using characteristic functions and the Cramér-Wold device [13, p. 170]. The argument is based on an application of the law of large numbers to the sequence σ_n, and is nearly verbatim the same as in the proof of Theorem 5.5. On the other hand, the tightness under P_ω of the sequence of processes J_n in $D[0,1]$ is evident from the criterion stated in Example 1 in [17, p. 336]. Notice that the criterion can be applied to J_n in virtue of (18). Once the weak convergence of J_n to a standard Brownian motion is proved under P_ω (for $P-$a.s. every environment ω), the same convergence under $\mathbb{P}$ follows from (3) and the bounded convergence theorem. □

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