Random linear recursions with Markov-dependent coefficients: regular variation in, regular variation out

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Abstract
We consider the equation $R_n = Q_n + M_n R_{n-1}$, with random Markov-dependent coefficients $(Q_n, M_n)_{n \in \mathbb{Z}} \in \mathbb{R}^2$, and show that the distribution tails of the stationary solution to this equation are regularly varying at infinity.

Keywords: stochastic difference equations, random linear recursions, heavy tails, regular variation, Markov models, chains of infinite order.

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1. Introduction and statement of results

Let $(Q_n, M_n)_{n \in \mathbb{Z}}$ be an ergodic stationary sequence of random pairs valued in $\mathbb{R}^2$ and

$$ R_n = Q_n + M_n R_{n-1}, \quad n \in \mathbb{N}, \; R_n \in \mathbb{R}. \tag{1} $$

This equation has a wide variety of real world and theoretical applications, see (Embrechts and Goldie, 1993; Vervaat, 1979). If $E (\log |M_0|) < 0$ and $E (\log^+ |Q_0|) < \infty$ (where $x^+$ denotes $\max\{x, 0\}$ for $x \in \mathbb{R}$), then for any $R_0$, the limit law of $R_n$ is that of $R = Q_0 + \sum_{n=1}^\infty Q_{-n} \prod_{i=0}^{n-1} M_{-i}$, and it is the unique initial distribution under which $(R_n)_{n \geq 0}$ is stationary (Brandt, 1986).

The distribution tails of $R$ were shown to be regularly varying in (Goldie, 1991; Grey, 1994; Grincevičius, 1975; Kesten, 1973) provided that $(Q_n, M_n)_{n \in \mathbb{Z}}$ form an i.i.d. sequence. Recall that $f : \mathbb{R} \to \mathbb{R}$ is called regularly varying if $f(t) = t^\alpha L(t)$ for some $\alpha \in \mathbb{R}$ and a slowly varying function $L(t)$, that is $L(\lambda t) \sim L(t)$ for all $\lambda > 0$. Here and henceforth $f(t) \sim g(t)$ (as a rule, we omit "as $t \to \infty$") means $\lim_{t \to \infty} f(t)/g(t) = 1$.

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The goal of this paper is to extend the results of (Grey, 1994; Grincevičius, 1975) to the case where \((Q_n, M_n)_{n \in \mathbb{Z}}\) are Markov-dependent. The extension is desirable in many, especially financial, applications. See (Perrakis and Henin, 1974; Collamore, 2009). We notice that an extension of the main result of (Kesten, 1973; Goldie, 1991) to a Markovian setup has been obtained in (de Saporta, 2005; Roitershtein, 2007), see also the companion paper (Ghosh et al., 2010). The mechanisms leading to regularly varying tails of \(R\) are quite different in (Kesten, 1973; Goldie, 1991) versus (Grey, 1994; Grincevičius, 1975). In particular, the latter case provides an instance of the phenomenon “regular variation in, regular variation out” for the model (1). This setup is particularly appealing because it enables one to gain insight into the asymptotic behavior of both the partial sums as well as extremes of random variables \(R_n\). see for instance (de Haan et al., 1989; Konstantinides and Mikosch, 2005; Rachev and Samorodnitsky, 1995; Rastegar et al., 2010).

**Assumption 1.1.** Let \((X_n)_{n \in \mathbb{Z}}\) be a stationary irreducible Markov chain on a countable state space \(\mathcal{D}\). \((Q_n, M_n)_{n \in \mathbb{Z}, n \in \mathcal{D}} \in \mathbb{R}^2\) be a sequence of independent random pairs such that for a fixed \(i \in \mathcal{D}\), \((Q_n, M_n)_{n \in \mathcal{D}}\) are i.i.d, and let

\[
Q_n = \sum_{j \in \mathcal{D}} Q_{n,j} \mathbf{1}_{\{X_n = j\}} = Q_n x_n \quad \text{and} \quad M_n = \sum_{j \in \mathcal{D}} M_{n,j} \mathbf{1}_{\{X_n = j\}} = M_n x_n.
\]

Furthermore, assume that there exists a constant \(\alpha > 0\) such that:

(A1) There is a slowly varying function \(L(t)\) and two sequences of constants \((q^{(n)}_i)_{i \in \mathcal{D}}\), \(\eta \in \{-1, 1\}\), such that for all \(n \in \mathbb{Z}\), \(\lim_{t \to \infty} \frac{P(Q_n, M_n > t)}{t^{-\alpha} L(t)} = q^{(n)}_i\) uniformly in \(i \in \mathcal{D}\). Moreover, \(\sum_{j \in \mathcal{D}} q^{(1)}_j > 0\) and \(\sup_{j \in \mathcal{D}} \max\{q^{(1)}_j, q^{(-1)}_j\} < \infty\).

(A2) There exists \(\beta > \alpha\) such that \(\sup_{i \in \mathcal{D}} \mathbb{E}(|M_0, i|^\beta) < \infty\).

(A3) For \(\eta \in \{-1, 1\}, i \in \mathcal{D}\), let \(m_i^{(\eta)} = \mathbb{E}(|M_0, i|^\alpha \mathbf{1}_{\{M_0, i, \eta > 0\}}\}, m_i = m_i^{(-1)} + m_i^{(1)}\). Then \(\sup_{i \in \mathcal{D}} m_i < 1\).

(A4) \(\lim_{t \to 0^+} P(M_{1,i} \leq \varepsilon) = P(M_{1,i} \leq 0)\) uniformly in \(i \in \mathcal{D}\).

Note that (2) defines a Hidden Markov Model (HMM), see for instance (Ephraim and Merhav, 2002) for a comprehensive survey of HMM and their applications in various areas. Heavy tailed HMM are considered for instance in (Resnick and Subramanian, 1998), see also references therein.

Let \(B_0 = \{x = (x_n)_{n \in \mathcal{D}} : \sup_{n \in \mathcal{D}} |x_n| < +\infty\} \subset \mathbb{R}^\mathcal{D}\) be equipped with the norm \(\|x\| = \sup_{n \in \mathcal{D}} |x_n|\). Then \(q^{(n)} = (q^{(n)}_i)_{i \in \mathcal{D}}, q = q^{(1)} + q^{(-1)}, m^{(n)} = (m_i^{(n)})_{i \in \mathcal{D}}\) and \(m = m^{(1)} + m^{(-1)}\) belong to \(B_0\). We write \(x > y\) (\(x < y\, \text{if} \, x \leq y, x \geq y\)) if the inequality holds for each component of \(x, y \in \mathbb{R}^\mathcal{D}\). We use alternatively \(x_i\) and \(x(i)\) to denote the \(i\)-th component of \((x_i)_{i \in \mathcal{D}}\). If \(X_n\) is a stationary irreducible Markov chain, let \(H\) be transition matrix of the backward chain \(X_{-n}\), that is \(H(i,j) = P(X_n = j | X_{n+1} = i)\). Define matrices \(G_{\eta}, \eta \in \{-1, 1\}, \text{and} \, G = G_{\eta}(i,j) = m_i^{(\eta)} H(i,j)\text{ and }G(i,j) = m_i H(i,j), i,j \in \mathcal{D}\). The spectral radius of \(G\) (as the operator in \(B_0\)) is less than one by (A3).

The following theorem shows that under Assumption 1.1, assuming in addition that \(P(M_0 > 0) = 1\), the upper tail of the distribution of \(R\) is regularly varying and has the same asymptotic structure as the upper tail of \(Q_0\).
Theorem 1.2. Let Assumptions 1.1 hold and suppose that $P(M_{0,i} > 0) = 1$ for all $i \in D$. Then, for all $i \in D$, $P(R > t|X_0 = i) \sim K_i t^{-\alpha} L(t)$, where $K = (K_i)_{i \in D} \in B_b$ is defined by $K = (I - G)^{-1}q^{(1)}$.

As in the case of i.i.d. coefficients $(Q_n, M_n)_{n \in \mathbb{Z}}$ (Grey, 1994; Grincevičius, 1975), Theorem 1.2 yields a similar result without the restriction $P(M_0 > 0) = 1$.

Theorem 1.3. Let Assumption 1.1 hold. Then, $P(R \cdot \eta > t|X_0 = i) \sim K_i^{(\eta)} t^{-\alpha} L(t)$ for all $\eta \in \{-1, 1\}$ and $i \in D$, where the vectors $K^{(\eta)} = (K_i^{(\eta)})_{i \in D} \in B_b$ are defined by

$$
K^{(\eta)} = \frac{1}{2} \left( (I - G)^{-1}q + \eta(I - G_+ + G_-)^{-1}(q^{(1)} - q^{(-1)}) \right) = (I - G_+ + G_-)^{-1}(q^{(\eta)} + G_-(I - G)^{-1}q).
$$

We remark that a broad class of non-Markov processes with “fading memory” $(X_n)_{n \in \mathbb{Z}}$ can be realized as functionals of a countable positive recurrent Markov chain. See for instance (Lalley, 1986; Berbee, 1987). Using the Markov representation, it is straightforward to extend Theorem 1.3 to $(Q_n, M_n)$ defined as in (2) by means of such processes.

The rest of the paper is organized as follows. Section 2 contains the proof of Theorem 1.2, Section 3 includes the proof of Theorem 1.3.

2. Proof of Theorem 1.2

Throughout this section, let Assumptions 1.1 hold and suppose in addition that for all $i \in D$, $P(M_{0,i} > 0) = 1$. The key to the result is Proposition 2.1 extending the corresponding statement in (Grincevičius, 1975; Grey, 1994).

Proposition 2.1. Let $Y$ be a random variable such that:

(i) $Y \in \sigma(X_n : n \leq 0) \cup \sigma(Q_{n,i}, M_{n,i} : n \leq 0, i \in D)$.

(ii) For $\eta \in \{-1, 1\}$ there exist non-negative constants $(c^{(\eta)}_i)_{i \in D}$ such that

(a) $\lim_{t \to -\infty} \frac{P(Y \wedge t|X_0 = i)}{t^{-\alpha} L(t)} = c^{(\eta)}_i$, uniform on $i \in D$.

(b) $\sup_{i \in D} c^{(\eta)}_i < \infty$.

Then $\lim_{t \to -\infty} \frac{P(Q_1 + M_i Y > t|X_0 = i)}{t^{-\alpha} L(t)} = q^{(1)}_i + m^{(1)}_i \sum_{j \in D} H(i, j)c^{(1)}_j$ uniformly in $i \in D$.

Proof. Let $(Y_i)_{i \in D}$ be a sequence of random variables independent of $(X_n)_{n \in \mathbb{Z}}$ and $(Q_{1,i}, M_{1,i})_{i \in D}$, such that $P(Y_i \leq t) = P(Y \leq t|X_0 = i)$ for all $t \in \mathbb{R}$ and $i \in D$. Since $P((Q_1, M_1, Y) \in \cdot|X_1 = i, X_0 = j) = P((Q_{1,i}, M_{1,i}, Y_j) \in \cdot)$ for all $i, j \in D$,

$$
P(Q_1 + M_1 Y > t|X_1 = i) = \sum_{j \in D} P(Q_1 + M_1 Y > t|X_1 = i, X_0 = j) H(i, j) = \sum_{j \in D} P(Q_{1,i} + M_{1,i} Y_j > t) H(i, j).
$$
By (Grey, 1994, Lemma 2), which is the i.i.d. prototype of our proposition,

\[ P(Q_{1,i} + M_{1,i}Y_j > t) \sim t^{-\alpha} L(t)(q_i^{(1)} + c_j^{(1)} m_i^{(1)}) \]  

(4)

To complete the proof of the proposition it suffices to show that the convergence in (4) is uniform in \( i \) and \( j \). Toward this end we decompose \( P(Q_{1,i} + M_{1,i}Y_j > t) \) into four individually tractable terms, as in (Grey, 1994, Lemma 2). Fix \( \varepsilon \in (0, 1) \) and write

\[
P(Q_{1,i} + M_{1,i}Y_j > t) = A_{ij}^{(1)}(t) - A_{ij}^{(2)}(t) + A_{ij}^{(3)}(t) + A_{ij}^{(4)}(t)\]

where

\[
A_{ij}^{(1)}(t) = P(Q_{1,i} > (1 + \varepsilon)t, Q_{1,i} + M_{1,i}Y_j \leq t),
\]

\[
A_{ij}^{(2)}(t) = P((1 - \varepsilon)t < Q_{1,i} \leq (1 + \varepsilon)t, Q_{1,i} + M_{1,i}Y_j \leq t),
\]

\[
A_{ij}^{(3)}(t) = P(Q_{1,i} > (1 - \varepsilon)t, Q_{1,i} + M_{1,i}Y_j > t),
\]

\[
A_{ij}^{(4)}(t) = P(Q_{1,i} \leq (1 - \varepsilon)t, Q_{1,i} + M_{1,i}Y_j > t).
\]

(A1) implies that \( A_{ij}^{(1)}(t) \) converges uniformly to \( q_i^{(1)}(1 + \varepsilon)^{-\alpha} \). For \( A_{ij}^{(2)} \) write

\[
A_{ij}^{(2)}(t) = P(Q_{1,i} > (1 + \varepsilon)t, Q_{1,i} + M_{1,i}Y_j \leq t; Y_j < -et^{\frac{\alpha}{2\alpha + \beta}})
\]

\[
+ P(Q_{1,i} > (1 + \varepsilon)t, Q_{1,i} + M_{1,i}Y_j \leq t; Y_j \geq -et^{\frac{\alpha}{2\alpha + \beta}})
\]

\[
\leq P(M_{1,i}Y_j \leq -et, Y_j \geq -et^{\frac{\alpha}{2\alpha + \beta}}) + P(Q_{1,i} > (1 + \varepsilon)t, Y_j < -et^{\frac{\alpha}{2\alpha + \beta}})
\]

\[
\leq P(M_{1,i} \geq t^{\frac{\alpha + \beta}{2\alpha}}) + P(Q_{1,i} > (1 + \varepsilon)t)P(Y_j < -et^{\frac{\alpha}{2\alpha + \beta}}).
\]

To obtain a bound on \( \frac{A_{ij}^{(2)}(t)}{t^{-\alpha} L(t)} \) which tends to zero uniformly in \( j \in D \) when \( t \to \infty \), we use assumption (A2) along with Chebyshev’s inequality \( P(M_{1,i} \geq t^{\frac{\alpha + \beta}{2\alpha}}) \leq E(M_{1,i}^{\beta}) \cdot t^{-\alpha + \beta} \) and the inequality \( P(Y_j < -et^{\frac{\alpha}{2\alpha + \beta}}) \leq C(\varepsilon t^{\frac{\alpha}{2\alpha + \beta}})^{-\alpha} L(t^{\frac{\alpha}{2\alpha + \beta}}) \), which is true for some \( C > 0 \) in virtue of condition (ii) of the proposition. Observe next that a uniform bound on \( \frac{A_{ij}^{(3)}(t)}{t^{-\alpha} L(t)} \) which tends to 0 as \( \varepsilon \to 0 \) follows directly from (A1). Finally, fix constants \( m \geq 1 \) and \( n \geq 1 \), and let \( A_{ij}^{(4)}(t) = A_{ij}^{(4,1)}(t) + A_{ij}^{(4,2)}(t) + A_{ij}^{(4,3)}(t) \), where

\[
A_{ij}^{(4,1)}(t) = \frac{P(Y_j > M_{1,i}^{-1}(t - Q_{1,i}))}{t^{-\alpha} L(t)} I_{\{Q_{1,i} \leq (1 - \varepsilon)t\}} I_{\{M_{1,i} > m\}}
\]

\[
A_{ij}^{(4,2)}(t) = \frac{P(Y_j > M_{1,i}^{-1}(t - Q_{1,i}))}{t^{-\alpha} L(t)} I_{\{Q_{1,i} \leq (1 - \varepsilon)t\}} I_{\{M_{1,i} \leq m\}} I_{\{|Q_{1,i}| > n\}}
\]

\[
A_{ij}^{(4,3)}(t) = \frac{P(Y_j > M_{1,i}^{-1}(t - Q_{1,i}))}{t^{-\alpha} L(t)} I_{\{Q_{1,i} \leq (1 - \varepsilon)t\}} I_{\{M_{1,i} \leq m\}} I_{\{|Q_{1,i}| \leq n\}}.
\]

Note that \( A_{ij}^{(4,1)}(t) \leq \frac{P(Y_j > M_{1,i}^{-1}; \varepsilon)}{t^{-\alpha} L(t)} \frac{(M_{1,i}^{-1}; \varepsilon)^{-\alpha} L(M_{1,i}^{-1}; \varepsilon)}{t^{-\alpha} L(t)} I_{\{M_{1,i} > m\}} \). Hence, by (ii)-

(b) of the proposition, \( A_{ij}^{(4,1)}(t) \leq \frac{C M_{1,i}^{\alpha} L(M_{1,i}^{-1}; \varepsilon)}{t^{-\alpha} L(t)} I_{\{M_{1,i} > m\}} \). By (Grey, 1994, Lemma 1), for any \( \delta > 0 \) there exists \( K = K(\delta) > 0 \) such that \( \sup_{t > 0} \frac{L(t)}{t^{\alpha}} \leq \max\{\lambda^\alpha, K \lambda^{-\delta}\} \) for
any $\lambda > 0$. Applying this with $\lambda = M_{i,j}^{-1} \epsilon$ and $\delta = \frac{\beta - \alpha}{\gamma} > 0$, we obtain

$$A_{i,j}^{(4,1)}(t) \leq E \left( \frac{CM_{i,j}^\alpha}{\epsilon^\alpha} \max\{(M_{i,j}^{-1} \epsilon)^\alpha, K(M_{i,j}^{-1} \epsilon)^{-\frac{\beta - \alpha}{\gamma}}\} I_{\{M_{i,j} > m\}} \right)$$

$$\leq E \left( \frac{CM_{i,j}^\alpha}{\epsilon^\alpha} (M_{i,j}^{-1} \epsilon)^\alpha + KM_{i,j}^{-\frac{\beta - \alpha}{\gamma}} \epsilon^{-\frac{\beta - \alpha}{\gamma}} \right) I_{\{M_{i,j} > m\}}).$$

Applying Hölder inequality with $p = \frac{2\beta}{\alpha + \beta}$, $q = \frac{2\beta}{\beta - \alpha}$ to the right-most expression gives:

$$A_{i,j}^{(4,1)}(t) \leq C \epsilon^{-\frac{2\beta}{\alpha + \beta}} E \left( (1 + KM_{i,j}^{\frac{\delta - \alpha}{\gamma}}) I_{\{M_{i,j} > m\}} \right)$$

$$\leq C \epsilon^{-\frac{2\beta}{\alpha + \beta}} E \left( (1 + KM_{i,j}^{\frac{\delta - \alpha}{\gamma}})^{\frac{\beta - \alpha}{\gamma}} P(M_{i,j} > m) \right)$$

where we used $(x + y)^p \leq 2^{p-1}(x^p + y^p)$ which is valid for $x, y \geq 0$ and $p > 1$. Since $P(M_{i,j} > m) \leq m^{-\beta} E(M_{i,j}^\beta)$, it follows from (A2) that $A_{i,j}^{(4,1)}(t)$ is uniformly bounded by a function of $m$ which tends to zero when $m \to \infty$. The term $A_{i,j}^{(4,2)}(t)$ is treated similarly, and we therefore omit the details. We next show the asymptotic of $A_{i,j}^{(4,3)}(t)$.

If $Q_{1,i} \leq (1 - \epsilon)t$ and $M_{i,j} \leq m$, then $M_{i,j}^{-1}(t - Q_{1,i}) \geq m^{-1}t$. Hence, in virtue of (A1) and condition (ii) of the proposition, we have $P - a.s.$, uniformly on $i,j \in D$,

$$P(Y_j > M_{i,j}^{-1}(t - Q_{1,i})) I_{\{Q_{1,i} \leq (1 - \epsilon)t\}} I_{\{M_{i,j} \leq m\}} \to_{t \to \infty} \epsilon_j^{(1)} I_{\{M_{i,j} \leq m\}}. \tag{5}$$

Furthermore, with probability one, uniformly on $i \in D$,

$$I^\alpha(M_{i,j}^{-1}(t - Q_{1,i}))^{-\alpha} I_{\{M_{i,j} \leq m\}} I_{\{|Q_{1,i}| \leq n\}} \to_{t \to \infty} M_{i,j}^{-\alpha} I_{\{M_{i,j} \leq m\}} I_{\{|Q_{1,i}| \leq n\}}. \tag{6}$$

Next, by Theorem 1.2.1 in (Bingham et al., 1987), $L(\lambda t)/L(t) \to_{t \to \infty} 1$ uniformly on compact $\lambda$-subsets of $(0, \infty)$. Hence, with probability one, uniformly on $i \in D$,

$$\frac{L(M_{i,j}^{-1}(t - Q_{1,i}))}{L(t)} I_{\{m^{-1} < M_{i,j} \leq m\}} I_{\{|Q_{1,i}| \leq n\}} \to_{t \to \infty} I_{\{m^{-1} < M_{i,j} \leq m\}} I_{\{|Q_{1,i}| \leq n\}}. \tag{7}$$

Finally, using Potter’s bounds on $L(\lambda t)/L(t)$ (see Theorem 1.5.6 in (Bingham et al., 1987)), we obtain that for any $\delta > 0$ there exists $t_0 = t_0(\delta)$ such that for all $t > t_0$,

$$\frac{L(M_{i,j}^{-1}(t - Q_{1,i}))}{L(t)} I_{\{M_{i,j} \leq m^{-1}\}} I_{\{|Q_{1,i}| \leq n\}} \leq \frac{\delta t}{t - n} I_{\{M_{i,j} \leq m^{-1}\}} I_{\{|Q_{1,i}| \leq n\}}. \tag{8}$$

Estimates (5)-(8) along with assumption (A4) and the bounded convergence theorem show that $\lim_{t \to \infty} A_{i,j}^{(4)}(t) = \epsilon_j^{(1)} E(M_{i,j}^\beta)$ uniformly on $i,j \in D$. □

To enable us to use Proposition 2.1 iteratively we need the following:
Lemma 2.2. Let $Y$ satisfy the conditions of Proposition 2.1, and let $\tilde{Y} = Q_1 + M_1 Y$. Then $\tilde{Y}$ satisfies condition (ii) of the proposition.

Proof. Apply Proposition 2.1 to $(Y, Q_1, M_1)$ and $(-Y, -Q_1, M_1)$. $\Box$

Following (Grey, 1994), we will bound the tail of $R$ by the tails of two sequences $R_n$ such that, respectively, $P(R_n > t | X_n = i) \geq P(R > t | X_0 = i)$ and $P(R_n > t | X_n = i) \leq P(R > t | X_0 = i)$. The first construction then implies $\limsup_{t \to \infty} P(R > t | X_0 = i) \leq K_i$, while the second yields $\liminf_{t \to \infty} P(R > t | X_0 = i) \geq K_i$.

Lemma 2.3. There exists a random variable $Z \geq 0$ satisfying the conditions of Proposition 2.1, such that $P(Q_1 + M_1 Z > t | X_0 = i) \leq P(Z > t | X_0 = i)$ for all $t > 0$, $i \in \mathcal{D}$.

Proof. One can use $Z$ defined in the proof of Lemma 3 of (Grey, 1994) provided that $c^*$ used in that construction satisfies condition $c^* \leq \max_{i \in \mathcal{D}}(1 - E(M_{0,i}^\alpha))^{-1}$. The latter maximum exists due to (A3) of Assumption 1.1. $\Box$

We are now in position to complete the proof of Theorem 1.2.

Lemma 2.4. For all $i \in \mathcal{D}$, $\limsup_{t \to \infty} \frac{P(R > t | X_0 = i)}{t^{-\alpha}L(t)} \leq (I - G)^{-1}q^{(1)}(i)$.

Proof. Let $R_0 = Z$, where $Z$ is introduced in Lemma 2.3. Then, in virtue of Lemma 2.3, $P(R_1 > t | X_1 = i) \leq P(R_0 > t | X_0 = i)$ for all $t > 0$ and $i \in \mathcal{D}$. This yields:

$$P(R_2 > t | X_2 = i) = \sum_{j \in \mathcal{D}} P(Q_2 + M_2 R_1 > t | X_2 = i, X_1 = j)H(i, j)$$

$$= \sum_{j \in \mathcal{D}} P(Q_2, i + M_2, R_1 > t | X_1 = j)H(i, j)$$

$$\leq \sum_{j \in \mathcal{D}} P(Q_1, i + M_1, R_0 | X_0 = j)H(i, j)$$

$$= P(Q_1 + M_1 R_0 > t | X_1 = i) = P(R_1 > t | X_1 = i).$$

Therefore $P(R_2 > t | X_2 = i) \leq P(R_1 > t | X_1 = i)$. Iterating the argument, we obtain

$$P(R_n > t | X_n = i) \leq P(R_{n-1} > t | X_{n-1} = i), \quad \forall n \in \mathbb{N}, i \in \mathcal{D}, t > 0. \tag{9}$$

Moreover, Proposition 2.1 and Lemma 2.2 imply that

$$P(R_n > t | X_n = i) \sim t^{-\alpha}L(t)[q^{(1)} + Gq^{(1)} + \ldots + G^{n-1}q^{(1)} + c^*G^n1](i),$$

where $c^* > 0$ is such that $P(Z > t) \sim c^* L(t) t^{-\alpha}$ and $1 \in \mathbb{R}^D$ has all components equal to 1. Since $P(Z > 0) = 1$, it follows from (9) that $P(R_n > t | X_n = i) \geq P(R > t | X_0 = i)$ for $n \geq 0$. Hence, $\limsup_{t \to \infty} \frac{P(R > t | X_0 = i)}{t^{-\alpha}L(t)} \geq (I - G)^{-1}q^{(1)}(i)$ for all $i \in \mathcal{D}$. $\Box$

Lemma 2.5. For all $i \in \mathcal{D}$, $\liminf_{t \to \infty} \frac{P(R > t | X_0 = i)}{t^{-\alpha}L(t)} \geq (I - G)^{-1}q^{(1)}(i)$. 

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Proof. Let $\mathcal{R} = Q_{-1} + M_{-1}Q_{-2} + M_{-1}M_{-2}Q_{-3} + \ldots$. Let $R_0 \geq 0$ be independent of $(X_n, Q_{1,n}, M_{1,n})_{n \geq 1 \in \mathbb{D}}$, such that $P(R_0 > t) = P(\mathcal{R} > 0, Q_0 > t)$ for all $t > 0$. Then 

$$
P(R_0 > t | X_0 = i) \leq P(Q_0 + M_0 \mathcal{R} > t | X_0 = i) = P(R > t | X_0 = i).$$

We will now use induction to show that for $n \geq 0$,

$$
P(R_n > t | X_n = i) \leq P(R > t | X_0 = i) \quad \forall \ t > 0, \ i \in \mathcal{D}. \tag{10}$$

Specifically, assuming (10) for some $n \geq 0$ we obtain

$$
P(R_{n+1} > t | X_{n+1} = i) = \sum_{j \in \mathcal{D}} P(R_{n+1} > t | X_{n+1} = i, X_n = j)H(i, j)$$

$$
= \sum_{j \in \mathcal{D}} P(Q_{n+1, i} + M_{n+1, i}R_n > t | X_n = j)H(i, j)
\leq \sum_{j \in \mathcal{D}} P(Q_{1,i} + M_1R > t | X_0 = j)H(i, j)
= P(Q_1 + M_1R > t | X_1 = i) = P(R > t | X_0 = i),
$$

Moreover, uniformly on $i \in \mathcal{D}$, 

$$
P(R_n > t | X_n = i) \sim t^{-a}L(t) \cdot [q^{(1)} + Gq^{(1)} + \ldots + G^{a-1}q^{(1)} + G^{a}\epsilon](i).
$$

This completes the proof of Lemma 2.5 in view of (10).

3. Proof of Theorem 1.3

The following result extends Lemma 4 of (Grey, 1994).

Lemma 3.1. Let $Y \in \sigma(X_n, Q_{n,i}, M_{n,i} : n \leq 0, i \in \mathcal{D})$ be a random variable such that

$$
c^{(n)}_i := \limsup_{t \to \infty} \frac{P(Y > t | X_{n,i} = \epsilon)}{t^{-a}L(t)} < \infty, \ d^{(n)}_i := \liminf_{t \to \infty} \frac{P(Y > t | X_{n,i} = \epsilon)}{t^{-a}L(t)} > -\infty, \ for \ all \ i \in \mathcal{D} \ and \ \eta \in \{-1, 1\}.
$$

Then for all $i \in \mathcal{D}$ and $\eta \in \{-1, 1\}$,

$$
\limsup_{t \to \infty} \frac{P((Q_1 + M_1Y) \cdot \eta > t | X_1 = i)}{t^{-a}L(t)} \leq q^{(n)}_i + \sum_{j \in \mathcal{D}} \sum_{\gamma \in \{-1, 1\}} G_\gamma(i, j)c^{(\gamma)}_j \quad \text{and}
$$

$$
\liminf_{t \to \infty} \frac{P((Q_1 + M_1Y) \cdot \eta > t | X_1 = i)}{t^{-a}L(t)} \geq q^{(n)}_i + \sum_{j \in \mathcal{D}} \sum_{\gamma \in \{-1, 1\}} G_\gamma(i, j)d^{(\gamma)}_j.
$$

Proof. Let $(Y_j)_{j \in \mathcal{D}}$ be a sequence of random variables independent of $(X_n)_{n \in \mathbb{D}}$ and $(Q_{1,i}, M_{1,i})_{i \in \mathcal{D}}$, such that $P(Y_j \cdot \eta > t) = P(Y \cdot \eta > t | X_0 = j)$ for $\eta \in \{-1, 1\}$. Then (3) implies $P((Q_1 + M_1Y) \cdot \eta > t | X_1 = i) = \sum_{j \in \mathcal{D}} P((Q_{1,i} + M_{1,i}Y_j) \cdot \eta > t)H(i, j)$. To complete, apply (Grey, 1994, Lemma 4) separately to each term $P((Q_{1,i} + M_{1,i}Y_j) \cdot \eta > t)$.

Let $R^* = |Q_0| + \sum_{n=1}^{\infty} |Q_{n-1}| \prod_{i=0}^{n-1} |M_{i-1}|$ be a stationary solution of the equation $R_{n+1} = |Q_n| + |M_n|R_n$. Notice that $-R^*$ is a stationary solution of the equation $R_{n+1} = -|Q_n| + |M_n|R_n$. Since $P(-R^* \leq R \leq R^*) = 1$, Theorem 1.2 ensures that Lemma 3.1 can be applied with $Y = R$. Let $a_i^{(n)} = \limsup_{t \to \infty} \frac{P(R > t | X_0 = i)}{t^{-a}L(t)}$.
\[ \liminf_{t \to \infty} \frac{P(R \geq t | X_0 = i)}{L(t)} \]. Denote \[ a^{(\eta)}_i = \left( a^{(\eta)}_i \right)_{i \in \mathbb{D}}, \quad b^{(\eta)}_i = \left( b^{(\eta)}_i \right)_{i \in \mathbb{D}} \] and \[ a = a^{(-1)} + a^{(1)}, \quad b = b^{(-1)} + b^{(1)}. \] The application of Lemma 3.1 yields \[ a \leq q + Ga \] and \[ b \geq q + Gb, \] which implies \[ (I - G)^{-1} q \leq b \leq a \leq (I - G)^{-1} q. \] This is only possible if \[ a^{(\eta)} = b^{(\eta)}, \] the inequalities in the conclusions of Lemma 3.1 are actually equalities, and thus \[ a^{(\eta)} = q^{(\eta)} + G_{\eta} a^{(\eta)} + G_{-\eta} a^{(-\eta)}, \] which implies the result of Theorem 1.3.

References


