Introduction to Diffusion Processes.

Arka P. Ghosh
Department of Statistics
Iowa State University
Ames, IA 50011-1210
apghosh@iastate.edu
(515) 294-7851.

February 1, 2010

Abstract

In this section we describe a class of stochastic processes called the diffusion processes. These are continuous-time, continuous state-space processes and their sample paths are everywhere continuous but nowhere differentiable. Since diffusions are defined through stochastic differential equations, we give a brief introduction to stochastic differential equations (which include a discussion on stochastic integration as well). Diffusion processes are defined in terms of these differential equations. Finally we conclude the section with some specific example of diffusion processes.

The history of diffusion processes begins with the botanist Brown, who in 1826-1827 observed that grains of pollen suspended in a water display a certain type of erratic motion, which did not fit any of the contemporary mathematical models. This motion came to be known as the Brownian motion. Einstein, in 1905, used physical principles to do mathematical analysis of this motion and Wiener provided a rigorous mathematical foundation for the Brownian motion (hence Brownian motion is also called the Wiener process). See Section 2.1.6.3 for more details on the Brownian motion. Diffusion processes, in some sense, are generalized versions of the Brownian motion. To fully understand the diffusion process, the reader should be familiar with the concept of stochastic calculus. We begin by introducing stochastic differential equations (SDEs) using heuristic arguments, followed by a brief introduction to stochastic calculus which is used to make the SDE formulation more rigorous. General diffusion process is defined next as solution to SDEs. Finally, we will discuss some major properties of diffusion processes and some common examples of diffusion processes.

Preliminaries:

It is common to characterize the evolution of a system through differential equations, which typically describes the rate of change of the system, in some ways, as functions of other variables (e.g. current value of the state, other variables etc.). Here, the state of the system can be, for example, the length of a queue in a (deterministic) model: Denoting the length of a queue at time \( t \geq 0 \) as \( q(t) \), one may have the following characterization of the evolution of the queue over time,

\[
q(0) = q_0, \quad \frac{d(q(t))}{dt} = \theta(t)q(t), \quad t \geq 0.
\]  (1)
Here, \( \theta(t) \) represents the rate of growth (or “decay”) of the queue-length at time \( t \geq 0 \), and \( q_0 \) is the initial queue-length at time \( t = 0 \). This “population-growth”-type model is quite common in many applications, and makes perfect sense if the growth parameter is a completely known function. However, in reality it is often not known completely, we may know that the rate function is “approximately” \( b(t) \) but it is reasonable to assume that it is subject to some random environmental effects, so that we have

\[
\theta(t) = b(t) + \text{“noise”},
\]

where we do not know the exact behavior of the noise term, but we know the its probability distribution. The following describes how (1) is dealt with, in presence of random noise.

**Heuristic formulation of SDEs:** One usually formulates (1) in this more general version:

\[
\frac{dQ(t)}{dt} = b(t, Q(t)) + \sigma(t, Q(t)) \cdot \text{“noise”},
\]

(2)

where \( b \) and \( \sigma \) are some given functions. For a good model for the “noise” part of the above equation, it is common to use a white-noise type model using the “increments” of the Brownian motion \( \{W(t)\} \) (see Section 2.1.6.3). Writing (2) in the increment form, one gets that for any \( 0 = t_0 < t_1 < \ldots < t_m = t \),

\[
(Q_{k+1} - Q_k) = b(t_k, Q_k)(\Delta t_k) + \sigma(t_k, Q_k)(\Delta W_k),
\]

(3)

where \( Q_j = Q(t_j), \Delta W_j = W(t_{j+1}) - W(t_j), \Delta t_k = t_{j+1} - t_j \). Thus, using (2), we get the following formulation describing the the dynamics of the process \( Q \), assuming \( Q(0) = q_0 \):

\[
Q(t) = q_0 + \sum_{k=1}^{m-1} b(t_k, Q_k)(\Delta t_k) + \sum_{k=1}^{m-1} \sigma(t_k, Q_k)(\Delta W_k).
\]

(4)

Now this formulation depends on the time-discretization, and it would be more acceptable in the limiting form as \( \Delta t_k \to \infty \), if the limit on the right hand side exists. This led to the formulation of stochastic integrals (which we describe below) and led to the following equation:

\[
Q(t) = q_0 + \int_0^t b(s, Q(s))ds + \int_0^t \sigma(s, Q(s))dW(s),
\]

(5)

Here the last integral is defined as the stochastic integral. The above equation is called the Stochastic Differential Equation (SDE) describing the dynamics of the process \( Q \). In addition to the above “integral form”, one often uses an equivalent “differential form” to describe the dynamics of (5):

\[
Q(0) = q_0, \quad dQ(t) = b(t, Q(t))dt + \sigma(t, Q(t))dW(t), \quad t \geq 0.
\]

(6)

A diffusion is the solution to the equation (5) (or (6)) described above, when it exists. Below, we give a brief introduction to the stochastic calculus used to define the stochastic integral used in these equations.

**Stochastic Integration:** Note that there are two types of integral expressions (involving some stochastic process, say \( \{X(s)\} \) and some function \( f(\cdot, \cdot) \)) in (5),

\[
\int_0^t f(s, X(s))ds, \quad \int_0^t f(s, X(s))dW(s).
\]
where $W$ is a Brownian Motion. In most situations, the first integral can be interpreted simply as the Riemann integral, in the classical way. There are several ways of interpreting the second integral, most of them involve viewing this integral as a suitable limit of the partial sum

$$\sum_{k=1}^{m} f(t_k, X(t_k))\Delta W_k,$$

as $\Delta t_j \to 0$ (here, $\{t_j\}$, $\Delta t_j$, $W_k$ is as defined in (3)). This leads to the most widely used version of stochastic integrals, called the Itô integral ([3] or [13] for more detail). This is the most common approach to define stochastic integrals, where the function-value at left-end-point of the intervals $[t_j, t_{j+1})$ is used to approximate the integrand. One popular alternative (where the middle point of the interval is used) to this is the Stratonovich Integral (see Chapter 3 of [13]).

**Itô Integrals:** Let $f(t, x)$ be a continuous function in $t$ and $x$, such that

$$\int_0^t E(f^2(u, X(u)))du < \infty.$$

Then the Ito integral of $f(t, X(t))$ with respect to $W(t)$ is defined as

$$\int_0^t f(s, X(s))dW(s) = \lim_{\Delta\rightarrow\infty} \sum_k \left(f(t_{k+1}, X_{k+1}) - f(t_k, X_k)\right)\Delta W_k,$$

(7)

where $\Delta_k = \max_k \Delta t_k$, and the limit is defined in the mean-squared sense (a random variable $V$ is called the mean square limit of a sequence $\{V_n\}$ if $\lim_n E[(V_n - V)^2] = 0$). For the above definition to make sense, we also need to have that the process $X$ is “adapted” to the filtration generated by the Brownian motion $W$. In simple words, if $\mathcal{F}_t$ denotes the sigma-algebra generated by $\{W(s) : s \leq t\}$ (it contains all the information of $W$-process till the time $t$), then $X(t)$ has to be measurable with respect to $\mathcal{F}_t$, for all $t \geq 0$, i.e $X(t)$ can be completely determined using the knowledge of $W(s)$ for $s \leq t$.

**Diffusion process:**

As defined earlier (see (5)-(6)), a diffusion process $\{X(t)\}$ is a solution to the following equation:

$$X(t) = x_0 + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s),$$

(8)

or, equivalently, to

$$X(0) = x_0, \quad dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t),$$

(9)

where $b(t, x)$ and $\sigma(t, x)$ are continuous functions of $t$ and $x$, such that

$$\int_0^t E(\sigma^2(s, X(s)))ds < \infty.$$

Here, $b(\cdot, \cdot)$ is called its drift function, and $\sigma(\cdot, \cdot)$ the diffusion function. Here $b$ and $\sigma$ can be thought of as the infinitesimal mean and variance of the increments of the process $X$ (see Section 2.1.6.2 for more detail).
First note that a diffusion process is defined as a solution to a (stochastic differential) equation, and we have to discuss the conditions under which the solution exists. But at this point, it is easy to consider some very basic examples:

**Example 1:** Setting \( x_0 = 0, b(t, x) = 0 \) and \( \sigma(t, x) = 1 \) yields that \( X(t) = W(t) \), the standard Brownian motion starting at the origin. Similarly, for constant coefficients \( b(x, t) = b, \sigma(x, t) = \sigma \) yields a Brownian motion with drift \( b \), diffusion parameter \( \sigma \). So, diffusion processes are more general than the Brownian motion processes described in Section 2.1.6.3.

The following result gives a general set of conditions under which diffusion exists (i.e there is a solution for the SDE in (8)):

**Existence conditions for diffusion processes:** Let \( T > 0 \) and \( b(\cdot, \cdot), \sigma(\cdot, \cdot) \) be measurable functions satisfying for all real \( x \), and for all \( t \in [0, T] \)

\[
|b(t, x)| + |\sigma(t, x)| < C(1 + |x|), \quad \text{and} \quad |b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| < D|x - y|,
\]

for some constants \( C > 0, D > 0 \), then there exists solution of the SDE in (8). In addition, this solution is also unique. Under much weaker conditions (e.g. continuity of \( b \) and \( \sigma \) strictly non-negative), one gets uniqueness in the sense that any two solutions will have the same probability distribution ("weak" uniqueness). See Chapter 5.2 of [13] for more discussion on this.

Example 1 described one of the simplest types of such processes and now we know conditions under which other diffusion processes exist. Now suppose \( X(t) \) is a diffusion process and \( Z(t) = g(X(t)) \) for a "nice" function \( g(\cdot) \). Can we characterize the dynamics of such a process \( \{Z(t)\} \)? The answer is provided by the celebrated Itô’s formula described below:

**Itô’s formula:** Let \( \{X(t)\} \) be a diffusion process (i.e. a solution of (8)). Let \( g(t, x) \) be a function that is continuously differentiable in \( t \) and twice continuously differentiable in \( x \). The stochastic process defined as \( Z(t) = g(t, X(t)) \) satisfies the following stochastic integral equation:

\[
Z(t) = Z(0) + \int_0^t \left( g_t(s, X(s)) + b(s, X(s))g_x(s, X(s)) + \frac{1}{2}\sigma^2(s, X(s))g_{xx}(s, X(s)) \right) ds
\]

\[
+ \int_0^t \sigma(s, X(s))g_x(s, X(s))dW(s), \quad (10)
\]

or, equivalently, the stochastic differential equation

\[
Z(0) = g(0, x), \quad dZ(t) = \left( g_t(t, X(t)) + b(t, X(t))g_x(t, X(t)) + \frac{1}{2}\sigma^2(t, X(t))g_{xx}(t, X(t)) \right) dt
\]

\[
+ \sigma(t, X(t))g_x(t, X(t))dW(t). \quad (11)
\]

Here \( g_t, g_x \) denote the partial derivatives of \( g \) with respect to \( t \) and \( x \), respectively and \( g_{xx} \) is the second order partial derivative with respect to \( x \).

The above formula is extremely important for dealing with diffusion processes and getting explicit solutions for them (by solving the defining SDEs as in (8)). The following give some applications of this formula, as well as some standard examples of diffusion processes.

**Examples.**

In Example 1, we already saw that Brownian motion is a simple example of a diffusion process. Here we discuss a few more standard examples.
Example 2: Geometric Brownian Motion process: Recall the motivating example at the beginning of this section (see (2)). In this situation, a queue-length process is can be modeled as the following diffusion process:

\[ Q(0) = q_0, \quad dQ(t) = bQ(t)dt + \sigma Q(t)dW(t), \]  

(12)

where \( b \) and \( \sigma > 0 \) are constants. To solve the above SDE, first we write it as follows:

\[ \frac{dQ(t)}{Q(t)} = bdt + \sigma dW(t). \]  

(13)

Next, applying Itô’s formula with \( g(t, x) = \ln(x) \) to the diffusion \( Q \) described in (12), we get (after simplification) that

\[ d\ln(Q(t)) = \frac{dQ(t)}{Q(t)} - \frac{1}{2}\sigma^2 dt. \]  

(14)

Using (13) and (14), we get that

\[ bdt + \sigma dW(t) = d\ln(Q(t)) - \frac{1}{2}\sigma^2 dt, \]

and hence, together with the fact that \( Q(0) = q_0 \), we have that

\[ Q(t) = q_0 \exp \left( (b - \sigma^2/2)t + \sigma W(t) \right). \]

This is the explicit expression of the diffusion process (implicitly) defined in (12). This diffusion process is known as the Geometric Brownian Motion process (with parameters \( b \) and \( \sigma \)).

Example 3: Ornstein-Uhlenbeck (OU) process: This is another common diffusion process defined by

\[ X(0) = x, \quad dX(t) = -bX(t)dt + \sigma dW(t), \]  

(15)

where \( b > 0 \) and \( \sigma > 0 \) are constants (here, \( b \) and \( \sigma \) are called the parameters of the OU process). One can show using calculations similar to the ones above that the explicit expression of this diffusion process is as follows:

\[ X(t) = \exp(-bt) \left( x + \sigma \int_0^t \exp(bs) dW(s) \right). \]  

(16)

As we mentioned earlier, diffusion equations are extremely important in operations research, especially in the context of modeling queues, storage processes etc. In continuous time models, it is often natural to use discrete-state processes (such as DTMC models in Section 2.1.2) because of discrete nature of the state-space (e.g. number of people in the queue). However, under suitable scalings (“diffusion”-scaling), the discrete state system can be approximated by a suitable diffusion. Often the physical models have natural constraints (such as queue-lengths being non-negative) which lead to constrained diffusions. Since there is a large literature on analysis of the diffusion processes, understanding the approximate system (driven by diffusions) is often possible and usually provides useful insights for the discrete-state physical systems. We provide a list of references below for further reading on these topics.
Further Reading.

More detail about diffusion processes can be found in classical textbooks on probability, such as [2]. For more recent developments, the we encourage the reader to consult [4], [5], [14], [15] (incomplete list). For students and researchers learning this material for the first time, some more accessible texts would be [3] and [13]. In many situations, diffusions are Markov processes and hence mathematical tools for Markov processes can be used for diffusions. A reference for general Markov processes would be [1].

Diffusion processes are often used in physics (see [7]), financial applications (see [10], [8]) and many other disciplines. For operations research applications, a general reference would be [6]. For more detailed reference for diffusion processes in queueing theory models, we recommend [12], [11] and [9].

References


