Heavy Traffic Analysis of a Simple Closed Loop Supply Chain

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April 8, 2010

Abstract

We consider a closed loop supply chain where new products are produced to order and returned products are refurbished for reselling. The solution to a price-setting problem enforces the "heavy traffic" condition, under which we address the production rate control problem for two types of cost functions. We solve a drift-control problem for an approximate system driven by a correlated two-dimensional Brownian motion. The solutions to this system are then used to obtain asymptotically optimal control policies. We also conduct a numerical study to explore the effects of different parameters on the optimal production rates and the resulting costs.

MSC2000 Subject Classification: Primary: 60K25, 60K20, 93E20; Secondary: 60K10, 60K30.
OR/MS subject classification: Primary: Queues - Diffusion Models, Queues - Limit Theorems, Queues - Networks, Queues - Optimization; Secondary: Inventory/Production - Policies - Marketing/pricing.
Running Title: Heavy Traffic Analysis of Closed Loop Supply Chain

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1 Introduction

We consider a queueing network model of a single firm that can control its production rate of new products but not their price in a competitive market. It produces new products to order. It allows customers to return some products after sale and refurbishes the returns for resale at a price that it chooses to balance the demands for new and refurbished products. The refurbished products are held in inventory. We assume that a customer is willing to wait while a new product is produced to her specifications, but potential buyers of refurbished products are impatient. While stylized, the model captures essential elements of a firm like Dell, Inc., which assembles new products to order, offers a generous return policy, and sells its stock of refurbished products in an online store. The relevant costs are associated with keeping customers waiting for new products, maintaining capacity to manufacture at a given rate, and losing potential sales of refurbished products. We derive an asymptotically optimal policy, which consists of the production rate for new products and the relative price of refurbished products, for this closed loop supply chain in heavy traffic.

Closed loop supply chains that encompass production, distribution, product returns, reprocessing and resale have gained increasing attention recently for both environmental and economic reasons. Reprocessing typically retains some of the value added by the original manufacturing process while preventing potentially harmful disposal and conserving both material and energy. To the original producer or a third party, reprocessing and reselling products can yield profits by reducing the cost of providing a functional product and expanding the market. The status of having been sold and returned may reduce the attractiveness of reprocessed products, yet a discounted price can create a lower-end market segment of consumers who are not willing to pay the full price for a new product but will accept a reprocessed one for a reduced price. This price should be low enough to make reprocessed products attractive compared with new ones and prevent their inventory from accumulating. On the other hand, too low a price for refurbished products could cannibalize the demand and profits earned by new products. Optimal pricing strategies for remanufactured goods have been analyzed in different contexts [8, 10, 11, 26]. Collecting or receiving and then refurbishing and reselling products introduces uncertainties in addition to those already present in manufacturing and selling new products. The availability of previously distributed products for refurbishment is subject to purchasers' decisions on whether and when to return them. Variability of the demand and product flows can create congestion or shortages that reduce the efficiency and economic viability of the closed loop supply chain. Queueing models have been employed in a number of studies to analyze the effectiveness of closed loop supply chain management policies under steady state conditions [14, 22, 30, 32], which imply non-negligible idle times in the service facilities. However, many managers recognize that idleness may reduce profit and prefer to utilize expensive processing resources as fully as possible by setting prices to increase demand. Such high utilization corresponds to heavy traffic in the queueing model. In recent years, several authors (e.g., [1, 2, 3, 6, 7, 12, 27, 29]) have employed heavy traffic approximations of various physical queueing networks and used techniques from stochastic control theory to obtain good queueing control policies.

We examine the two decision variables, price and production rate, sequentially. First, we formulate a price-setting problem (also known as the static planning problem) to maximize profit in the fluid-scaled system. The optimal solution naturally imposes the heavy traffic conditions (see [29] for a similar analysis). The heavy traffic conditions require that the arrival rates and the service rates of the queueing system are "balanced" in some sense. We show that a solution to a profit maximization problem for the fluid scaled queueing system (the so-called static planning problem) naturally imposes the heavy traffic assumption on our model. Intuitively, this can be interpreted as follows: if the manufacturer decides to maximize profit based on the average behavior of the system, the optimal prices will enforce that the arrivals (functions of the price-variable) match the services and, hence, satisfy the heavy traffic conditions. Second, under the heavy traffic conditions, we solve the problem of finding an optimal production rate to minimize an appropriate cost function. Such heavy traffic analyses often follow a sequence of steps outlined by Harrison [17] (see also [5]), which involves solving a diffusion control problem (called the Brownian Control Problem or the BCP) that approximates the queueing control problem and then interpreting its solution to obtain meaningful control policies for the original queueing control problem. In this paper, we consider two common forms of cost functionals: long-run average (ergodic) cost and the infinite horizon discounted cost. For each of these cost functions, we carry out the analysis following Harrison's scheme: we first formulate and solve the BCP; then we propose a candidate for optimal control policy for the queueing model by interpreting the solution of the BCP; and finally, we prove the asymptotic optimality of the proposed policy using weak convergence methods. We also discuss some comparative statics and carry out a numerical study to explore the effect of system parameters on the optimal production rates and resulting costs. The main contributions of this paper are the following: This is the first paper to our knowledge that successfully applies heavy traffic machinery to optimize performance of a closed loop supply chain. A natural price setting problem is shown to enforce heavy traffic conditions in such a supply chain. This paper also provides complete heavy traffic analysis to obtain optimal production rates under the two most common cost functions in the control literature. Despite the existence of a large literature for heavy traffic analysis of queueing networks, most articles with such provably optimal solutions focus on one-dimensional problems. There are very few such complete analyses for two-dimensional models prior to this one (see [3, 7]). This article provides one such analysis for a two-dimensional model where the associated diffusion model is driven by a two-dimensional correlated Brownian motions. Having solved the diffusion control problem, we establish the main asymptotic optimality results using properties of an appropriate Skorohod map (regulator map) and weak convergence techniques.

The rest of the paper is organized as follows: In Section 2, we describe the model and the control problems in detail. Next, we discuss the static planning problem and the heavy traffic conditions for our queueing network. Our main theorems (Theorems 2.7 and 2.8) describing asymptotically optimal policies are also stated in this section. In Section 3, we address the two BCPs for two different choices of the cost function. Section 4 contains weak convergence analysis to prove the main results. Section 5 contains some comparative statics and numerical analysis of the two cost problems. Finally in Section 6, we summarize the paper, provide a comparison of our results with the steady-state analysis of similar models under the average cost functional and conclude with possible extensions to this work. An Appendix contains proofs of some of the more standard results that are used in our analysis.

2 Problem Description

We study a simple model of a closed loop supply chain in which a producer manufactures new products to order. Some new products are returned by the customers after evaluation. We assume that any new product may be returned after sale with probability $\beta \in (0, 1)$. These returned products can no longer be sold as new. Instead, they are inspected, refurbished and placed into inventory to be resold (see Figure 1). As in Vorasayan and Ryan [33], we assume the producer is a price-taker in the market for new products, whose exogenously-determined price is p_N, normalized so that $0 < p_N < 1$. It sets the price for refurbished products, p_R, such that $p_R < p_N$. Consumer (normalized) valuation of new products, denoted as p, is uniformly distributed on $(0, 1)$. A consumer who is willing to pay a price p for a new product is willing to pay at most δp for a refurbished product, where $0 < \delta < 1$. Given the prices, the consumer chooses between new and refurbished products to maximize his/her surplus: $\max\{p - p_N, \delta p - p_R, 0\}$. If $p_R \geq \delta p_N$ then $\delta p - p_R < p - p_N$ for any $p < 1$; therefore, we assume $p_R < \delta p_N$ to guarantee some demand for refurbished products. Likewise, we assume $p_R \geq p_N - (1 - \delta)$ because otherwise, $p - p_N < \delta p - p_R$ for any $p < 1$ and no demand would exist for new products. A strategic decision variable for the producer is

$$\rho \equiv \frac{p_R}{p_N}, \tag{2.1}$$

such that $\rho \in (1 - \frac{1-\delta}{p_N}, \delta)$. In terms of this price ratio, the normalized demand rate for new products represents the proportion of a fixed number of customers per unit time who will buy the new product, i.e., those for whom $p > p_N$ and $p - p_N > \delta p - \rho p_N$, and is given by:

$$\lambda_N(\rho) = 1 - \frac{p_N(1 - \rho)}{1 - \delta}. \tag{2.2}$$

The corresponding demand rate for refurbished products is

$$\lambda_R(\rho) = \frac{p_N}{1 - \delta}\left(1 - \frac{\rho}{\delta}\right), \tag{2.3}$$

which represents the proportion of customers for whom $\delta p > p_R$ and $p - p_N < \delta p - \rho p_N$.

In our model, the demands for new and refurbished products follow Poisson processes with the rates $\lambda_N(\rho)$ and $\lambda_R(\rho)$ for a chosen value of ρ. These and other parameters are constant over an implicit study horizon represented by the model, which is reasonable for a product category such as business laptop, but not intended for specific models within that category. We assume that the time required to produce a new product is exponentially distributed with rate $\mu > 0$ and that the manufacturing server is not allowed to idle unless the queue of new product orders is empty. When a demand for a refurbished item arrives, if such a product is available in inventory then the demand is satisfied; otherwise, the customer is lost. Let $X_1(t)$ denote the length of the new product customer queue and $X_2(t)$ denote the number of refurbished products in inventory at time t. Then, given $X_i(0) = x_i$, $i = 1, 2$, we model X_1 and X_2 as:

$$X_1(t) = x_1 + N_1(\lambda_N(\rho)t) - N_2\left(\int_0^t \mu\, 1_{\{X_1(s)>0\}} ds\right), \tag{2.4}$$

$$X_2(t) = x_2 + \Phi\left[N_2\left(\int_0^t \mu\, 1_{\{X_1(s)>0\}} ds\right)\right] - N_3\left(\int_0^t \lambda_R(\rho)\, 1_{\{X_2(s)>0\}} ds\right), \tag{2.5}$$

where $N_i(\cdot), i = 1, 2, 3$, are independent unit Poisson processes. For any nonnegative integer m, $\Phi(m) = \sum_{k=1}^m \phi_k$, where $\{\phi_k\}$ is a sequence of i.i.d Bernoulli(β) random variables. Here and for the rest of the paper, 1_A, will denote the indicator function of a Borel set A (i.e. $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$, if $x \notin A$). In the above display, $\Phi(m)$ represents the (random) number of products that are returned by customers out of the first m purchased products. See Chapter 6 of [24] to see a more general construction of jump-Markov process, with state space $\mathbb{Z}$, as a

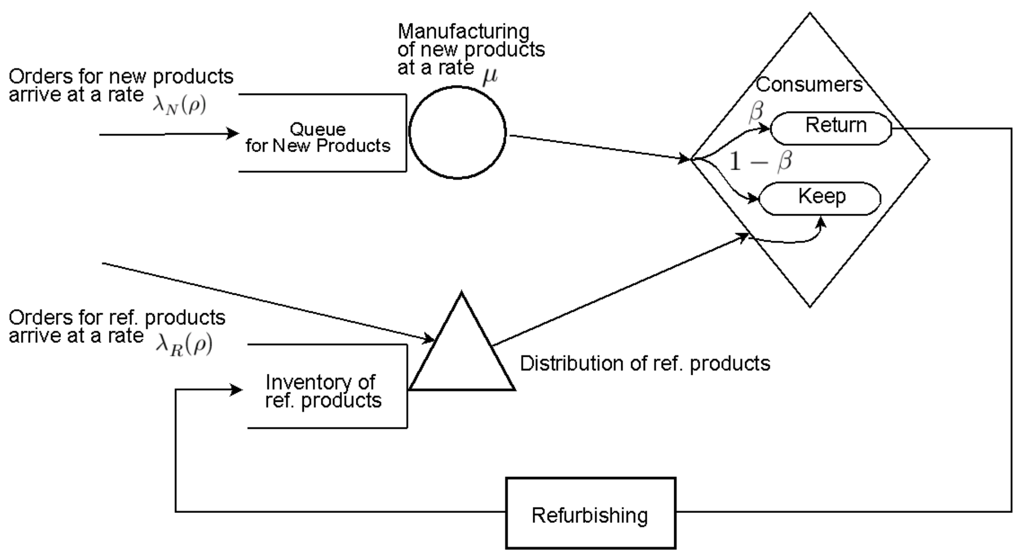

Figure 1: Closed loop supply chain network

linear combination of time changed versions of unit Poisson process as in (2.4) and (2.5). We assume that all returned products are refurbished and that return, if it occurs, and refurbishment are both instantaneous. This assumption approximates the situation where returns are mainly due to buyer remorse or unmet expectations rather than any real defect, so that refurbishment amounts to inspection or testing and repackaging. In Section 6 we describe extensions to incorporate exponentially-distributed delays in return and/or refurbishment as well as disposal of some fraction of returns. Define processes L_1 and L_2 as follows:

$$L_1(t) = \mu \int_0^t 1_{\{X_1(s)=0\}} ds, \quad L_2(t) = \lambda_R(\rho) \int_0^t 1_{\{X_2(s)=0\}} ds. \tag{2.6}$$

The parameter μ represents the average number of new products that can be manufactured per unit time. The process $L_1(\cdot)$ is defined as μ multiplied by the time that the manufacturing server has idled so far. In that sense, $L_1(t)$ represents the average number of new-product customers that could have been served in the interval $[0, t]$ during the server's idle time when no new-product customers are waiting. Using a similar interpretation, $L_2(t)$ captures the average number of lost sales of refurbished products. We assume that the cost for storing refurbished products is mainly fixed with respect to quantity, and therefore little affected by policies that influence $X_2(t)$. Ideally, we prefer policies which produce fewer lost sales of refurbished goods. This preference is reflected in the definitions of the cost functionals in (2.12)-(2.13) below (see Section 6 for possible extensions of the model).

Our goal in this paper is to optimize (1) the price of refurbished products relative to new products and (2) the production rate of new products. We carry out this optimization in two steps. First, we solve a static planning problem in terms of the fluid-scaled processes and the long term average demand rates, which are assumed to satisfy (2.2) and (2.3). The profit is maximized by setting the price ratio and long term average production rate so the system is in heavy traffic. Second, we carry out a heavy traffic analysis of the system, and find an asymptotically optimal service rate under optimal prices.

As it is commonly done for such analysis, we will consider a sequence of networks (indexed by a parameter n), each having the same structure, but the parameters of the n-th network depend on the index n, and we will require that as $n \to \infty$, the system achieves heavy traffic (see Assumption 2.1 and 2.4 below). A physical network that is close to being in heavy-traffic can be thought of as one element of this sequence with a large value of n. Hence, from now on, we will consider a sequence of networks indexed by n and all the processes and parameters depend on n (denoted by a superscript n, e.g., $\lambda_N^n(\rho), X_1^n(t)$, etc.). We assume that ρ^n does not depend on n (i.e., $\rho^n \equiv \rho$), and its optimal value will be determined by the limit behavior of the system (in the static planning problem). Note that, since this queueing model is a Jackson network, the queue lengths of each network in the sequence can be analyzed exactly in steady-state. In Section 5, we illustrate how the result of such a prelimit analysis coincides with the asymptotic analysis for a special case of the long-run average cost function. However, the discounted cost depends on the transient behavior of the queue lengths, and we lack any exact characterization of the arrival process to the second queue during the transient phase.

A *policy* consists of the price ratio $\rho \in (0,1)$ as defined in (2.1) and the manufacturing rate sequence $\{\mu^n\}$. We assume the following basic convergence properties for the parameters of this model:

Assumption 2.1 *There exist* $\theta_i \in I\!R$, $i = 1,2,3$, $\{\bar\lambda_N(\rho) > 0, \rho \in [0,1]\}$, $\{\bar\lambda_R(\rho) > 0, \rho \in [0,1]\}$, $\bar\mu > 0$, *and* $x_1, x_2 \geq 0$ *such that*

$$(i) \quad \sqrt{n}(\lambda_N^n(\rho) - \bar\lambda_N(\rho)) \to \theta_2, \sqrt{n}(\lambda_R^n(\rho) - \bar\lambda_R(\rho)) \to \theta_3 \text{ for all } \rho \in [0,1], \tag{2.7}$$

$$(ii) \quad \sqrt{n}(\mu^n - \bar\mu) \to \theta_1, \tag{2.8}$$

$$(iii) \quad \beta\theta_2 = \theta_3 \quad \text{and} \quad \hat x_i^n = x_i^n/\sqrt{n} \to x_i \text{ as } n \to \infty, i = 1,2. \tag{2.9}$$

Remark 2.2 *The assumptions in (2.7) state that there are long-run average rates (for arrivals) to which the parameters of the n-th system converge. They also specify that this convergence takes place at the rate of* $\frac{\theta_i}{\sqrt{n}}$, *where* $\theta_i, i = 2,3$, *are the convergence rates, which is the natural rate of convergence for heavy traffic assumption of diffusion scaled systems. It has been shown (for control problems involving linear holding costs) that for any admissible policy* $\{\mu^n\}$ *that produces finite asymptotic costs (see* (2.12) *and* (2.13) *below), (2.8) holds (see [34]). So in our analysis, we restrict consideration to admissible controls that satisfy (2.8). The first part of (2.9) is a technical assumption that reduces the problem dimension: because of this, the limiting diffusion control problem is effectively one-dimensional. To be more specific, the terms* u^n *and* $\tilde u^n$ *in (4.53)-(4.54) converge to the same constant* u *(using (4.50)) which is the drift parameter governing both the* *processes in the limiting model (3.19). Existence of such asymptotic limits is a standard assumption in heavy traffic analysis.*

We will carry out the asymptotic analysis of the diffusion scaled queueing model. Therefore, we need to define the diffusion scale before introducing the cost functional. The analysis also involves the so-called fluid-scaled processes. For any process $\psi^n(\cdot)$ described here, $\bar{\psi}^n(\cdot)$ and $\hat{\psi}^n(\cdot)$ will denote the fluid- and diffusion-scaled processes respectively, given by:

$$\bar{\psi}^n(t) = \frac{\psi^n(nt)}{n}, \; \hat{\psi}^n(t) = \frac{\psi^n(nt)}{\sqrt{n}}, \text{ for all } t \geq 0, n = 1, 2, \ldots. \tag{2.10}$$

In this paper, we analyze two types of cost functionals: the long-run average cost (also known as the "ergodic cost") and the infinite horizon discounted cost, each of which involves the following components: a control cost for the service rate, a backorder cost for new products, and a linear cost per lost customer of refurbished products. Here we assume that the inventory of the refurbished products does not incur any variable cost for the manufacturer; hence, the cost functionals include no holding cost for the refurbished products. These components of costs are given in terms of functions $c(\cdot), h(\cdot)$ and a constant penalty rate k, which satisfy the following assumptions.

Assumption 2.3 *The functions $c(\cdot)$ and $h(\cdot)$ are nonnegative, continuous, nondecreasing and convex on $[0, \infty)$ and k is a positive constant. Also, $c(x) = 0$, for $x \leq 0$ and there exist K and $N > 0$ such that $h(\cdot)$ satisfies $0 \leq h(x) \leq K(1 + x^N)$ for all $x \geq 0$.*

Let

$$u^n(\rho) = \sqrt{n}(\mu^n - \lambda_R^n(\rho)/\beta), \quad \rho \in [0, 1]. \tag{2.11}$$

Under the heavy traffic conditions described in Section 2.1, the quantity $(\beta\mu^n - \lambda_R^n(\rho))$ can be thought of as the net inventory growth rate for the refurbished products in the n-th system. This quantity tends to zero at the rate $\sqrt{n}$ as n approaches infinity, as shown in (4.50)) in Section 4. Here, u^n defined in (2.11) captures this rate. As argued in Remark 3.1, the limit of u^n is assumed to be non-negative. Hence, for simplicity, we only allow non-negative values for $u^n(\rho)$ for all $n \geq 1$

and $\rho \in [0,1]$.

The long-run average cost is given by :

$$\begin{aligned}\hat{I}_0(x_1,x_2,\rho,\{\mu^n\}) &\doteq \liminf_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}E\int_0^T \left[\left(c\left(u^n(\rho)\right)+h(\hat{X}_1^n(s))\right)ds + k\, d\hat{L}_2^n(s)\right] \\ &= \liminf_{n\to\infty}\limsup_{T\to\infty}\left[c\left(u^n(\rho)\right)+\frac{1}{T}E\int_0^T h(\hat{X}_1^n(s))ds + k\,\frac{1}{T}E\hat{L}_2^n(T)\right]. \quad (2.12)\end{aligned}$$

For a fixed discount rate $\alpha > 0$, the infinite horizon discounted cost is given by:

$$\begin{aligned}\hat{J}_0(x_1,x_2,\rho,\{\mu^n\}) &\doteq \liminf_{n\to\infty} E\int_0^\infty e^{-\alpha s}\left[\left(c\left(u^n(\rho)\right)+\left(\hat{X}_1^n(s)\right)^2\right)ds + k\, d\hat{L}_2^n(s)\right] \\ &= \liminf_{n\to\infty}\frac{c\left(u^n(\rho)\right)}{\alpha} + E\int_0^\infty e^{-\alpha s}\left[\left(\hat{X}_1^n(s)\right)^2 ds + k\, d\hat{L}_2^n(s)\right]. \quad (2.13)\end{aligned}$$

In this paper we solve the infinite horizon discounted cost problem with backorder cost $h(x) = x^2$. The term $c(u^n(\rho))$ in (2.12) and (2.13) represents the cost of choosing production rate μ relative to the arrival rate for refurbished products (suitably scaled), while the term $h(\hat{X}_1^n(s))$ is the cost per backorder (of new items) per unit time. The infinitesimal quantity, $k\, d\hat{L}_2^n(s)$, is the penalty for lost sales of refurbished items. Here x_1 and x_2 are the (asymptotic) initial lengths of the backorders of new items and inventory of refurbished items as defined in (2.9). Note that the choice of u^n uniquely determines the service rate sequence μ^n. Since $c(x) = 0$ for all $x \le 0$, for each $n \ge 1$ the control cost can be thought of as $\tilde{c}(\mu^n) \doteq c(u^n(\rho)) = c(\sqrt{n}(\mu^n - \lambda_R^n(\rho)/\beta))$, which is an increasing function of μ^n, for each fixed $\lambda_R^n(\rho)$. Prior to analysis, for both control problems with costs (2.12) and (2.13), it is not clear which among the three components of the cost is dominant.

Even if the cost functions involve only diffusion-scaled processes, to be able to carry out the analysis one needs to have the fluid system "stable." Hence, we define the static planning problem below, and deduce the conditions for "heavy traffic." The process of solving the static planning problem also solves the problem of price setting (i.e., choosing ρ).

2.1 Static Planning Problem

Static planning problems are formulated by constructing a system where the fluid-scaled processes are replaced by their long-run averages (or fluid limits) and solving a suitable optimization problem involving those averages (see [19, 25, 28, 29]). In the fluid limit, we formulate a deterministic problem to choose ρ and $\bar{\mu}$ that maximize the profit rate subject to stability conditions on both queues. The profit consists of revenue from the sale of both new and refurbished returns less the cost per unit time associated with producing new products at rate $\bar{\mu}$. Let $\gamma(\cdot)$ be a nondecreasing function. The profit maximization problem is:

$$\begin{aligned} \max_{\rho,\bar{\mu}} \quad & p_N(1-\beta)\bar{\lambda}_N(\rho) + p_R\beta\bar{\lambda}_N(\rho) - \gamma(\bar{\mu}) \\ \text{s.t.} \quad & \bar{\lambda}_N(\rho) \leq \bar{\mu} \\ & \beta\bar{\lambda}_N(\rho) \leq \bar{\lambda}_R(\rho). \end{aligned}$$

The first term in the objective function is revenue per unit time from the sale of new products (less a refund for returned products), whose sales are limited by demand. The second term is revenue per unit time from the sale of refurbished products, whose sales are limited by supply. The first constraint ensures that supplies of new products are sufficient to meet the demand on average. The second constraint restricts the supply of refurbished products to not exceed their demand; otherwise, inventories of refurbished products would accumulate without bound.

The objective is separable into its revenue and cost components, where revenue depends only on ρ and cost depends only on $\bar{\mu}$. Clearly, the optimal limiting production rate equals its lower bound: $\bar{\mu}^* = \bar{\lambda}_N(\rho^*)$, where ρ^* is the optimal price ratio. The revenue rate is proportional to:

$$(1-\beta+\beta\rho)\bar{\lambda}_N(\rho) = (1-\beta+\beta\rho)(1-\frac{p_N}{1-\delta}+\frac{p_N}{1-\delta}\rho),$$

which is a convex quadratic function of ρ that is minimized by

$$\tilde{\rho} = \frac{-p_N(1-2\beta)-\beta(1-\delta)}{2\beta p_N}.$$

The largest feasible value for ρ is found uniquely by solving the second constraint as an equality:

$$\rho' = \frac{\delta\left[p_N(1+\beta)-\beta(1-\delta)\right]}{p_N(1+\delta\beta)}.$$

It is easy to verify that $\rho'-\tilde{\rho} \geq 0$ for any $p_N > 0$ if $\beta < \frac{1}{2}$ (in fact, $\tilde{\rho} < 0$ in this case). Therefore, revenue is an increasing function of feasible ρ, so that $\rho^* = \rho'$. The unique solution to the static planning problem is given by $(\rho^*, \bar{\mu}^*)$ such that $\bar{\lambda}_N(\rho^*) = \frac{1}{\beta}\bar{\lambda}_R(\rho^*)$ and $\bar{\mu^*} = \bar{\lambda}_N(\rho^*)$. This relation constitutes the heavy traffic condition, and for the rest of the paper, we will assume that all admissible policies satisfy this condition, in addition to Assumption 2.1. See Figure 2.

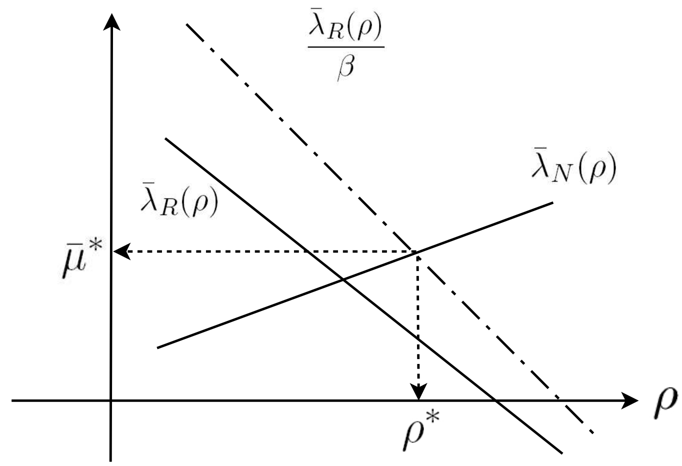

Figure 2: Price determination from static planning problem.

Assumption 2.4 (Heavy Traffic) *Any admissible policy satisfies*

$$\bar{\mu}^* = \bar{\lambda}_N(\rho^*) = \frac{1}{\beta}\bar{\lambda}_R(\rho^*).$$

There are different equivalent methods of arriving at the above described heavy traffic assumption needed for analyzing diffusion-scaled systems (see [5, 17]), and it can be verified that those methods also yield the same heavy traffic condition as we have here. For example, one conventional way for defining heavy traffic (see [18]) is to require that the following holds: There exists a unique optimal solution $(\tilde{r}^*, \tilde{x}^*)$ satisfying $\tilde{r}^* = 1$ and $A\tilde{x}^* = \mathbf{1}$ to the following linear program,

$$\text{minimize} \quad r \quad \text{subject to} \quad R\tilde{x} = \alpha, \ A\tilde{x} \leq r\mathbf{1} \ \textit{and} \ \tilde{x} \geq 0.$$

Here the decision variables $\tilde{x}$ represent average rates at which activities are undertaken and the objective is a vector of upper bounds on the utilization rates for processing resources; the constants R, A and α are related to the parameters of the network: The average rates of arrival to the two servers from outside the system are given by α (note that in our formulation of the inventory process dynamics, λ_R serves as the service rate and not an external arrival rate), while the input-output matrix R and capacity-consumption matrix A are defined as

$$\alpha = \begin{pmatrix} \bar{\lambda}_N(\rho) \\ 0 \end{pmatrix}, \; R = \begin{pmatrix} \bar{\mu} & 0 \\ -\beta\bar{\mu} & \bar{\lambda}_R(\rho) \end{pmatrix}, \; A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

See [18] for more details on this formulation of heavy traffic. It is easy to verify that the $(\rho^*, \bar{\mu}^*)$ satisfying the conditions in Assumption 2.4 also satisfies the heavy traffic condition as defined in [18].

Remark 2.5 *The heavy traffic conditions along with (2.2) and (2.3) applied to the fluid limit imply that*

$$\bar{\mu}^* = \bar{\lambda}_N(\rho^*) = \frac{1 - p_N}{1 + \beta\delta} = \frac{1}{\beta}\bar{\lambda}_R(\rho^*).$$

The assumption $1 - \frac{1-\delta}{p_N} < \rho^* = \rho' < \delta$ *holds true for any* $p_N < 1$.

Definition 2.6 (Two queueing control problems) *Under Assumptions 2.1 and 2.4, the price variable is set as* ρ^**, which determines the demand rates of new and refurbished products, as well as the long-run average service rate* $\bar{\mu}^*$*. A sequence of service rates* $\{\mu^n\}$ *is said to be admissible if it satisfies Assumptions 2.1 and 2.4 with* ρ^* *and* $\bar{\mu}^*$*.*

The first queueing control problem is to find an asymptotically optimal service rate sequence $\{\mu_n^*\}$ *that minimizes*

$$\hat{I}(x_1, x_2, \{\mu^n\}) \doteq \hat{I}_0(x_1, x_2, \rho^*, \{\mu^n\}) \tag{2.14}$$

over all admissible controls $\{\mu^n\}$*.*

The second queueing control problem is to find an asymptotically optimal service rate sequence $\{\mu_n^*\}$ *that minimizes*

$$\hat{J}(x_1, x_2, \{\mu^n\}) \doteq \hat{J}_0(x_1, x_2, \rho^*, \{\mu^n\}) \tag{2.15}$$

over all admissible controls $\{\mu^n\}$*.*

The following are the two main theorems of this article, which show the existence of optimal controls for two queueing control problems described in Definition 2.6.

Theorem 2.7 *There exists a* $u_a^* \geq 0$ *such that*

$$\mu_a^{n,*} = \frac{1}{\beta}\lambda_R^n(\rho^*) + \frac{u_a^*}{\sqrt{n}},\ n = 1, 2, \ldots \tag{2.16}$$

is an asymptotically optimal sequence of service rates for the first queueing control problem defined in Definition 2.6. Furthermore, this u_a^* *satisfies the following:*

$$u_a^* = \text{argmin}_{u \geq 0} \int_0^\infty e^{-y}\Big[c(u) + h\left(\frac{{\sigma_1}^2}{2u}y\right)\Big]dy,$$

where $\sigma_1^2 = \bar{\lambda}_N + \bar{\mu}$.

Theorem 2.8 *There exists a* $u_d^* \geq 0$ *such that*

$$\mu_d^{n,*} = \frac{1}{\beta}\lambda_R^n(\rho^*) + \frac{u_d^*}{\sqrt{n}},\ n = 1, 2, \ldots \tag{2.17}$$

is an asymptotically optimal sequence of service rates for the second queueing control problem defined in Definition 2.6.

The existence of u_a^* is proved in Theorem 3.4, and that of u_d^* is established in Theorem 3.14. Note that, in each of the control problems, the choices in (2.16) and (2.17) are not unique. For example, for the first problem,

$$\tilde{\mu}_a^{n,*} = \lambda_N^n(\rho^*) + \frac{u_a^*}{\sqrt{n}},\ n = 1, 2, \ldots \tag{2.18}$$

is another such choice. Note that these two choices are asymptotically equivalent, in the sense that the behavior of the diffusion-scaled system under these two choices are the same (as a consequence of the fact that u^n and $\tilde{u}^n$ defined in (4.50) are asymptotically equivalent).

3 Brownian Control Problems

The Brownian control problem (BCP) for a queueing network is formulated by replacing the linear combination of centered processes in the scaled queue equations (see the martingale terms $\hat{W}_i^n$ for $i = 1, 2$ defined in Section 4) by suitable Brownian motions, and constructing a diffusion control problem ([5, 16] etc.). The solutions to such control problems often contain useful insights about the queueing control problems, and are commonly used in such analysis.

In the next section, we will establish that the sequence $(\hat{X}_1^n, \hat{X}_2^n)$ converges weakly to a two-dimensional process (X_1, X_2), which is a reflecting diffusion with state space in the first quadrant of $\mathbb{R}^2$. Furthermore, (X_1, X_2) satisfies the following stochastic differential equations:

$$\begin{aligned} X_1(t) &= x_1 - ut + \sigma_1 W_1(t) + L_1(t) \\ X_2(t) &= x_2 + \beta ut + \sigma_2 W_2(t) - \beta L_1(t) + L_2(t), \end{aligned} \tag{3.19}$$

where $W_1(\cdot)$ and $W_2(\cdot)$ are two standard Brownian motion processes and they are correlated. Their dependence is described by $E[W_1(t)W_2(t)] = -rt$, where $r = \frac{\bar{\mu}\beta}{\sigma_1\sigma_2}$ and the constants σ_1 and σ_2 are given by $\sigma_1 = \sqrt{\bar{\lambda}_N + \bar{\mu}}$ and $\sigma_2 = \sqrt{\bar{\lambda}_R + \bar{\mu}\beta(1-\beta)}$. The local-time processes L_1 and L_2 are non-decreasing and satisfy $L_1(0) = L_2(0) = 0$. Furthermore,

$$\int_0^t 1_{[X_1(s)>0]} dL_1(s) = 0 \text{ and } \int_0^t 1_{[X_2(s)>0]} dL_2(s) = 0, \text{ for all } t > 0.$$

The local time processes L_1 and L_2 keep the state processes, respectively X_1 and X_2, non-negative. $X_1(t)$ represents the limiting queue length for the new product at time t and $X_2(t)$ represents the limiting inventory of the refurbished products at time $t > 0$.

Remark 3.1 *The constant $u \geq 0$ in (3.19) is the control parameter which captures the rate at which the manufacturing rate (of new products) deviates under diffusion scaling from the ones specified by heavy traffic. The non-negativity of u guarantees the finiteness of the cost functional in (3.20), since otherwise X_1 in (3.19) will be transient leading to infinite holding costs. Hence, we focus only on controls $u \geq 0$ throughout the article.*

The stochastic system described in (3.19) is known as a “Brownian control system.” In the following two subsections, we will consider this system under the two types of cost structures, and solve the corresponding control problems in each case.

3.1 BCP with long-run average cost

First we describe the long-term average cost structure associated with the first control problem. For such a Brownian control system described by (3.19), we consider the long term expected average cost function given by

$$I(x_1, x_2, u) = \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T (c(u) + h(X_1(s)))ds + k\ L_2(T)\right]. \tag{3.20}$$

Recall that $c(u)$ represents the control cost, $h(X_1(t))$ represents the holding cost for queue length $X_1(t)$ and the constant $k > 0$ represents the penalty per lost customer for refurbished products. Since $c(u)$ is time independent, the cost function in (3.20) can be written as

$$I(x_1, x_2, u) = c(u) + \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T h(X_1(s))ds + k\ dL_2(t)\right]. \tag{3.21}$$

In the following discussion, we intend to obtain an optimal control $u^* \geq 0$ that minimizes $I(x_1, x_2, u)$ over all constant controls $u \geq 0$. We can represent the value function of this stochastic control problem by

$$V(x_1, x_2) = \inf_{u\geq 0} I(x_1, x_2, u).$$

Next, using the equation for $X_1(\cdot)$, we intend to compute $\lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T h(X_1(s))ds\right]$ explicitly. First we introduce the constant $\gamma(u)$ for each control $u \geq 0$ by

$$\gamma(u) = \frac{2u}{\sigma_1^2} \int_0^\infty h(x) e^{-\frac{2ux}{\sigma_1^2}} dx. \tag{3.22}$$

Since $h(\cdot)$ has polynomially-bounded growth (see Assumption 2.3), this constant $\gamma(u)$ is finite for each $u \geq 0$. The following proposition connects a part of the cost in (3.21) with $\gamma(u)$. The proof of this result involves application of the Itô's formula, and is somewhat standard. Hence, we state this as a proposition here without proof, and describe a proof in the Appendix.

Proposition 3.2 *Let* (X_1, X_2) *satisfy (3.19). Then* $\lim_{T\to\infty} \frac{1}{T} E \int_0^T h(X_1(s))ds$ *exists and equals* $\gamma(u)$.

In the next lemma, we examine how $E[L_2(T)]$ grows as a function of T. Since $u > 0$ in (3.19), $X_1(\cdot)$ is a reflecting diffusion process with constant negative drift $-u$. It has a stationary distribution with exponentially decaying tail and $\lim_{T\to\infty} \frac{1}{T}(L_1(T) - uT) = 0$ a.s. (see [15], page 29). Therefore, the magnitude of the term $(ut - L_1(t))$ must be at most of the order $\sup_{0\le s\le t} |W_1(s)|$. Hence using the equation for $X_2(\cdot)$ in (3.19), we expect that the process $X_2(\cdot)$ does not reach the origin that often. Consequently, $L_2(t)$ increases at a much slower rate than $L_1(t)$, as t tends to infinity. Intuitively, in the absence of a variable inventory cost for refurbished products, the cost function emphasizes customer service with the result that lost sales of refurbished products are negligible in the limit. In the following lemma, we verify this intuition by proving that $\lim_{T\to\infty} \frac{L_2(T)}{T} = 0$ a.s. as well as in L^1.

Lemma 3.3 *Let* L_2 *be the local time process of* X_2 *in (3.19). Then*

$$\lim_{T\to\infty} \frac{L_2(T)}{T} = 0 \ \textit{a.s. and} \ \lim_{T\to\infty} \frac{E[L_2(T)]}{T} = 0.$$

Proof. L_1 has the representation (see [15])

$$L_1(t) = \max\left\{0, -\inf_{0\le s\le t}(x_1 + \sigma_1 W_1(s) - us)\right\}. \tag{3.23}$$

Consider the Brownian motion $W_1(\cdot)$ and the maximum process M_1 defined by $M_1(t) = \sup_{0\le s\le t} |W_1(s)|$. Then (3.23) implies that $L_1(t) - ut \le \sigma_1 M_1(t)$, for all $t \ge 0$. Similarly L_2 has the representation

$$L_2(t) = \max\left\{0, -\inf_{0\le s\le t}(x_2 + \sigma_2 W_2(s) + \beta us - \beta L_1(s))\right\}. \tag{3.24}$$

Again we introduce $M_2(t) = \sup_{0\le s\le t} |W_2(s)|$. Notice that

$$-\inf_{0\le s\le t}\left[x_2 + \sigma_2 W_2(s) + \beta us - \beta L_1(s)\right] = \sup_{0\le s\le t}\left[\beta(L_1(s) - us) - \sigma_2 W_2(s) - x_2\right].$$

Using the estimate $L_1(t) - ut \leq \sigma_1 M_1(t)$ for all $t \geq 0$, we obtain

$$\beta(L_1(s) - us) - \sigma_2 W_2(s) - x_2 \;\leq\; \beta\sigma_1 M_1(s) + \sigma_2 M_2(s) \;\leq\; \beta\sigma_1 M_1(t) + \sigma_2 M_2(t), \quad \text{for all } 0 \leq s \leq t.$$

Also, $\beta\sigma_1 M_1(t) + \sigma_2 M_2(t) \geq 0$ for all $t \geq 0$. Therefore, it follows that

$$\max\left\{0, \sup_{0\leq s\leq t} [\beta(L_1(s) - us) - \sigma_2 W_2(s) - x_2]\right\} \leq \beta\sigma_1 M_1(t) + \sigma_2 M_2(t), \text{ for all } t \geq 0.$$

Hence by (3.24), we obtain that $0 \leq L_2(t) \leq \sigma_2 M_2(t) + \beta\sigma_1 M_1(t)$, for all $t > 0$. By the properties of the maximum process of Brownian motion (see pages 95 and 112 of [21]), we know that $\lim_{T\to\infty} \frac{M_1(T)}{T} = \lim_{T\to\infty} \frac{M_2(T)}{T} = 0$ a.s., and $E[M_i(t)] \leq C\sqrt{T}$ for $i = 1, 2$, where C is a constant. Therefore, it follows that

$$\lim_{T\to\infty} \frac{L_2(T)}{T} = 0 \text{ a.s. and } \quad \lim_{T\to\infty} \frac{E[L_2(T)]}{T} = 0.$$

■

Using Proposition 3.2 and Lemma 3.3, we can provide the following explicit representation of the cost function $I(\cdot)$ in (3.21):

$$I(x_1, x_2, u) = c(u) + \gamma(u) = c(u) + \frac{2u}{\sigma_1^2} \int_0^\infty h(x) e^{-\frac{2ux}{\sigma_1^2}}\, dx. \tag{3.25}$$

This expression can be simplified to obtain

$$I(x_1, x_2, u) = \int_0^\infty e^{-y} \Big[c(u) + h\left(\frac{\sigma_1^2}{2u} y\right)\Big] dy. \tag{3.26}$$

The above computations establish the following theorem.

Theorem 3.4 *The value function $V(x_1, x_2)$ of the long-run average cost problem described in (3.20)-(3.21) is independent of (x_1, x_2) and it has the following representation*

$$V \equiv V(x_1, x_2) = \inf_{u\geq 0} \int_0^\infty e^{-y} \Big[c(u) + h\left(\frac{\sigma_1^2}{2u} y\right)\Big] dy.$$

Furthermore, for

$$F(u) = \int_0^\infty e^{-y}\Big[c(u) + h\left(\frac{\sigma_1^2}{2u}y\right)\Big]dy, \tag{3.27}$$

an optimal control $u_a^* > 0$ *is given by* $F(u_a^*) = \min_{u\geq 0} F(u)$.

To compute u_a^* we differentiate the above function of u to obtain

$$F'(u) = c'(u) - \frac{\sigma_1^2}{2u^2}\int_0^\infty e^{-y} y h'\left(\frac{\sigma_1^2}{2u}y\right) dy.$$

To find a candidate for u_a^*, we let $F'(u) = 0$, which yields the following necessary condition:

$$2(u_a^*)^2 c'(u_a^*) = \sigma_1^2 \int_0^\infty e^{-y} y h'\left(\frac{\sigma_1^2}{2u_a^*}y\right) dy.$$

In the case where both $c(\cdot)$ and $h(\cdot)$ are convex twice differentiable increasing functions, the above condition is also sufficient, because

$$F''(u) = c''(u) + \frac{\sigma_1^4}{2u^4}\int_0^\infty e^{-y} y^2 h''\left(\frac{\sigma_1^2}{2u}y\right) dy + \frac{\sigma_1^2}{u^3}\int_0^\infty e^{-y} y\, h'\left(\frac{\sigma_1^2}{2u}y\right) dy > 0.$$

For example, when $c(x) = x^m$ and $h(x) = x^q$, where $m \geq 1$ and $q \geq 1$, we obtain

$$u_a^* = \left(\frac{q}{m} q! \left(\frac{\sigma_1^2}{2}\right)^q\right)^{\frac{1}{m+q}}. \tag{3.28}$$

3.2 BCP with infinite horizon discounted cost

In the previous section, we have noticed that the expected cost $k\ E[L_2(T)]$ which represents the penalty incurred from lost customers for refurbished products during the time interval $[0, T]$ grows at a rate much slower than T as T tends to infinity. In fact, $\lim\limits_{T\to\infty} \frac{E[L_2(T)]}{T}$ is equal to zero. For this reason, the optimal control policy developed in the previous section is not influenced by this cost component. To capture the effect of the penalty incurred from lost customers for refurbished products, we also consider an infinite horizon discounted cost structure for the same model in (3.19). In this case, the cost functional as well as the optimal policy are affected by the cost component corresponding to the lost customers for refurbished products as well as by the initial data x_1 and

x_2 of (3.19).

In our analysis of this cost structure, we use $h(x) = x^2$ to perform explicit computations. A main difficulty in our analysis is to obtain an explicit formula for $E[L_2(T)]$ in this two-dimensional model described in (3.19). For this reason, we are able to establish a nontrivial optimal control $u_d^* > 0$ for the discounted cost only when the initial data (x_1, x_2) belong to a certain region in $\mathbb{R}^2$.

Here we analyze the infinite horizon discounted cost structure given by

$$J(x_1, x_2, u) = E \int_0^\infty e^{-\alpha t}[(c(u) + X_1(t)^2)dt + k\ dL_2(t)], \tag{3.29}$$

where $\alpha > 0$ and $k > 0$ are positive constants. We can rewrite this cost functional in the form

$$J(x_1, x_2, u) = \frac{c(u)}{\alpha} + \Phi(x_1, u) + \Psi(x_1, x_2, u), \tag{3.30}$$

where

$$\Phi(x_1, u) = E \int_0^\infty e^{-\alpha t} X_1(t)^2 dt, \tag{3.31}$$

and

$$\Psi(x_1, x_2, u) = k\ E \int_0^\infty e^{-\alpha t} dL_2(t). \tag{3.32}$$

The value function for this control problem is given by

$$Q(x_1, x_2) = \inf_{u \geq 0} J(x_1, x_2, u). \tag{3.33}$$

In the following lemma, for a given control $u \geq 0$, we compute $\Phi(x_1, u)$ described in (3.31). The proof of the lemma is given in the Appendix.

Lemma 3.5 *Let $\Phi(x_1, u)$ be defined by (3.31). Then*

$$\Phi(x_1, u) = \frac{1}{\alpha}\left[\left(x_1 - \frac{u}{\alpha}\right)^2 + \left(\frac{\sigma_1^2}{\alpha} + \frac{u^2}{\alpha^2}\right)\right] - \frac{2u}{\alpha^2 \lambda_1(u)} e^{-\lambda_1(u) x_1}, \tag{3.34}$$

where $\lambda_1(u) = \frac{1}{\sigma_1^2}\left(\sqrt{(u^2 + 2\alpha\sigma_1^2)} - u\right)$.

For our two-dimensional model described in (3.19), next consider the functional Ψ given in

(3.32). Here we are unable to compute $\Psi(x_1, x_2, u)$ explicitly. Therefore, we obtain an upper bound for the quantity $\Psi(x_1, x_2, u) - \Psi(x_1, x_2, 0)$ in the next lemma. Here, $\Psi(x_1, x_2, 0)$ represents the cost defined by (3.32) in the case of zero control. To identify the dependence of the processes on the control $u \geq 0$, we rewrite our model equation (3.19) in the following form:

$$\begin{aligned} X_1^u(t) &= x_1 - ut + \sigma_1 W_1(t) + L_1^u(t) \\ X_2^u(t) &= x_2 + \beta ut + \sigma_2 W_2(t) - \beta L_1^u(t) + L_2^u(t), \end{aligned} \tag{3.35}$$

where L_1^u and L_2^u are local time processes of X_1^u and X_2^u respectively. Next we introduce the process $\tilde{X}^u$ by

$$\tilde{X}^u(t) = x_2 + \beta ut + \sigma_2 W_2(t) + \tilde{L}^u(t), \tag{3.36}$$

where $\tilde{L}^u(t)$ is the local time process of $\tilde{X}^u$ at the origin and hence $\tilde{L}^u(t) \geq 0$ for all $t \geq 0$. Notice that (3.36) can be rewritten as

$$\tilde{X}^u(t) = x_2 + \beta ut + \sigma_2 W_2(t) - \beta L_1^u(t) + \left(\beta L_1^u(t) + \tilde{L}^u(t)\right). \tag{3.37}$$

We compare (3.37) with the second equation in (3.35). Recall that the local time process L_2^u is the minimal continuous non-decreasing process which keeps the sum $(x_2 + \beta ut + \sigma_2 W_2(t) - \beta L_1^u(t) + L_2^u(t))$ non-negative. But in (3.37), $\tilde{X}^u(t) \geq 0$ for all t and therefore we obtain the inequality

$$L_2^u(t) \leq \beta L_1^u(t) + \tilde{L}^u(t), \quad \text{for all } t \geq 0. \tag{3.38}$$

This estimate of $L_2^u(t)$ will be useful in the next proposition.

Proposition 3.6 *Let the initial value (x_1, x_2) be fixed. Then the cost functional $J(x_1, x_2, u)$ defined in (3.29) is continuous in the control variable $u \geq 0$.*

Proof. For $J(x_1, x_2, u)$, we consider the representation (3.30). The function $c(u)$ is continuous in u and by lemma 3.5, $\Phi(x, u)$ is also continuous in u. Therefore, it remains to show that $\Psi(x_1, x_2, u)$ is continuous in the variable u.

For any $u \geq 0$, by (3.32) and Fubini's theorem, we obtain

$$\begin{aligned}\Psi(x_1, x_2, u) &= k\ E\left[\int_{t=0}^{\infty}\left(\int_{s=t}^{\infty}\alpha e^{-\alpha s}ds\right)dL_2^u(t)\right] \\ &= k\ E\left[\int_{s=0}^{\infty}\alpha e^{-\alpha s}L_2^u(s)ds\right].\end{aligned}$$

Therefore

$$\Psi(x_1, x_2, u) = \alpha\ k\ E\int_{t=0}^{\infty} e^{-\alpha t}L_2^u(t)dt = \alpha\ k\int_{t=0}^{\infty} e^{-\alpha t}E[L_2^u(t)]dt. \tag{3.39}$$

Next we fix $u \geq 0$ and let $\{u_n\}$ converge to u. We assume that $0 \leq u_n \leq K$ for some fixed constant $K > 0$. It suffices to show that $\lim_{u_n \to u} \Psi(x_1, x_2, u_n) = \Psi(x_1, x_2, u)$. For each $u \geq 0$, the local time process $L_1^u(t)$ has the representation

$$L_1^u(t) = \max\left\{0, \sup_{0\leq s\leq t}(us - \sigma_1 W_1(s) - x_1)\right\}, \tag{3.40}$$

and therefore, for each $T > 0$ it is evident that $L_1^{u_n}(t)$ converges to $L_1^u(t)$ uniformly on $[0, T]$.

Next, $L_2^u(t)$ has the representation

$$L_2^u(t) = \max\left\{0, \sup_{0\leq s\leq t}(\beta L_1^u(s) - \beta us - \sigma_2 W_2(s) - x_2)\right\}. \tag{3.41}$$

Since $u_n \to u$ and $L_1^{u_n}(t)$ and $L_2^{u_n}(t)$ converge uniformly to $L_1^u(t)$ and $L_2^u(t)$, respectively, for all $0 \leq t \leq T$, from the above representation (3.40)-(3.41). For each u_n, $0 \leq L_1^{u_n}(t) \leq x_1 + Kt + \sigma_1 \sup_{0\leq s\leq t}|W_1(s)|$ and $0 \leq L_1^{u_n}(t) \leq x_2 + |\beta|Kt + |\beta|L_1^{u_n}(t) + \sigma_2 \sup_{0\leq s\leq t}|W_2(s)|$. Now let $M(t)$ be the process defined by

$$M(t) = |\beta|x_1 + x_2 + 2|\beta|Kt + |\beta|\sigma_1 \sup_{0\leq s\leq t}|W_1(s)| + \sigma_2 \sup_{0\leq s\leq t}|W_2(s)|.$$

Using Doob's inequality we obtain $E[M(t)^2] \leq C_o(1 + t^2)$, where $C_o > 0$ is a generic constant. Hence $E[M(t)] \leq \sqrt{C_o}(1 + t)$ and $E\int_0^\infty e^{-\alpha t}M(t)dt < \infty$. Since $0 \leq L_2^{u_n}(t) \leq M(t)$ and $L_2^{u_n}(t)$ converges to $L_2^u(t)$ a.s. as u_n tends to u, we can apply the Dominated Convergence Theorem to conclude that

$$\lim_{n\to\infty} E\int_0^\infty e^{-\alpha t}L^{u_n}(t)dt = E\int_0^\infty e^{-\alpha t}L^u(t)dt.$$

Hence, $\lim_{n\to\infty} \Psi(x_1, x_2, u_n) = \Psi(x_1, x_2, u)$ and this completes the proof. ∎

Remark 3.7 *Since* $\lim_{u\to\infty} c(u) = +\infty$ *and* $J(x_1, x_2, u) > \frac{c(u)}{\alpha}$*, the above proposition guarantees the existence of a non-negative optimal control* u_d^**.*

Next, we obtain some sufficient conditions that guarantee a strictly positive optimal control u^*, which leads to a non-trivial asymptotically optimal sequence of controls for the original sequence of controlled queueing systems.

Lemma 3.8 *Let* $\Psi(x_1, x_2, u)$ *and* $\Psi(x_1, x_2, 0)$ *be as described in (3.32). Then*

$$\Psi(x_1, x_2, u) - \Psi(x_1, x_2, 0) \leq \alpha\,\beta\,k \int_0^\infty e^{-\alpha t}(E[L_1^u(t)] - E[L_1^0(t)])dt + k\,\alpha \int_0^\infty e^{-\alpha t} E[\tilde{L}^u(t)]dt + kx_2, \tag{3.42}$$

where L_1^u*,* L_1^0 *and* $\tilde{L}^u$ *are the local time processes described in (3.35).*

Proof. Using (3.39), we obtain

$$\Psi(x_1, x_2, u) - \Psi(x_1, x_2, 0) = k\alpha \int_{t=0}^\infty e^{-\alpha t}(E[L_2^u(t)] - E[L_2^0(t)])dt. \tag{3.43}$$

Next we estimate $(E[L_2^u(t)] - E[L_2^0(t)])$ for each $t \geq 0$. By (3.38), we have $E[L_2^u(t)] \leq \beta E[L_1^u(t)] + E[\tilde{L}^u(t)]$. On the other hand, using the second equation of (3.35), we have $E[L_2^0(t)] - \beta E[L_1^0(t)] + x_2 = E[X_2^0(t)] \geq 0$, for all $t \geq 0$. Therefore, $E[L_2^0(t)] \geq \beta E[L_1^0(t)] - x_2$ for all $t \geq 0$. Consequently,

$$E[L_2^u(t)] - E[L_2^0(t)] \leq \beta \left[E[L_1^u(t)] - E[L_1^0(t)]\right] + E[\tilde{L}^u(t)] + x_2 \text{ for all } t \geq 0.$$

Thus, from this estimate in (3.43) we have (3.42). ∎

In the next lemma, we compute the integrals in (3.42).

Lemma 3.9

(i) For each $u \geq 0$,

$$\alpha \int_0^\infty e^{-\alpha t} E[L_1^u(t)]dt = \frac{1}{2\alpha}\left(\sqrt{u^2 + 2\alpha\sigma_1^2} + u\right) e^{-\lambda_1(u)x_1}, \tag{3.44}$$

where $\lambda_1(u) = \frac{1}{\sigma_1^2}\left(\sqrt{u^2 + 2\alpha\sigma_1^2} - u\right)$.

(ii) For each $u \geq 0$,

$$\alpha \int_0^\infty e^{-\alpha t} E[\tilde{L}^u(t)]dt = \frac{1}{2\alpha}\left(\sqrt{\beta^2 u^2 + 2\alpha\sigma_2^2} - \beta u\right) e^{-\lambda_2(u)x_2}, \tag{3.45}$$

where $\lambda_2(u) = \frac{1}{\sigma_2^2}\left(\sqrt{\beta^2 u^2 + 2\alpha\sigma_2^2} + \beta u\right)$.

Proof. First notice that $\alpha \int_0^\infty e^{-\alpha t} E[L_1^u(t)]dt = E\int_0^\infty e^{-\alpha t} dL_1^u(t)$. Let $Q(x) = e^{-\lambda_1(u)x}$, where $\lambda_1(u) = \frac{1}{\sigma_1^2}\left(\sqrt{u^2 + 2\alpha\sigma_1^2} - u\right)$. Then Q satisfies

$$\frac{\sigma_1^2}{2} Q''(x) - uQ'(x) - \alpha Q(x) = 0, \quad \text{for } x > 0 \ \text{ and } Q'(0) = -\lambda_1(u).$$

Next, we consider the first equation of (3.35) and apply Itô's lemma to $Q(X_1^u(t))e^{-\alpha t}$ to obtain

$$E[Q(X_1^u(T))]e^{-\alpha T} = Q(x_1) - \lambda_1(u) E \int_0^T e^{-\alpha t} dL_1^u(t).$$

We let T tend to $+\infty$ and obtain

$$Q(x_1) = \lambda_1(u) E \int_0^\infty e^{-\alpha t} dL_1^u(t).$$

Hence (3.44) follows. The proof of (3.45) follows essentially along the same steps by using $Q(x) = e^{-\lambda_2(u)x}$ where $\lambda_2(u)$ is given in (3.45) and the process $\tilde{X}$ in (3.36).

■

The following proposition follows from Lemmas 3.8 and 3.9.

Proposition 3.10

$$\begin{aligned}\Psi(x_1,x_2,u)-\Psi(x_1,x_2,0) &\leq \frac{k\beta}{2\alpha}\left[\left(\sqrt{u^2+2\alpha\sigma_1^2}+u\right)e^{-\lambda_1(u)x_1}-\sqrt{2\alpha\sigma_1^2}e^{-\frac{\sqrt{2\alpha}}{\sigma_1}x_1}\right]\\ &+ \frac{k}{2\alpha}\left[\left(\sqrt{\beta^2u^2+2\alpha\sigma_2^2}-\beta u\right)e^{-\lambda_2(u)x_2}\right]+kx_2. \qquad (3.46)\end{aligned}$$

The proof of this proposition is straightforward and therefore omitted.

Remark 3.11 *The estimates we have obtained in the proof of the above proposition also yield the following upper bound of the cost functional $J(x_1,x_2,u)$ defined in (3.30):*

$$J(x_1,x_2,u)<\frac{c(u)}{\alpha}+\Phi(x_1,u)+k\left[\frac{\beta}{\lambda_1(u)}e^{-\lambda_1(u)x_1}+\frac{1}{\lambda_2(u)}e^{-\lambda_2(u)x_2}\right]+kx_2. \qquad (3.47)$$

In the next proposition, we obtain a sufficient condition which guarantees $J(x_1,x_2,u)<J(x_1,x_2,0)$ where the cost functional J is defined in (3.30).

Proposition 3.12 *Let (x_1,x_2) be the initial data in (3.19). If there is a control $u\geq 0$ that satisfies*

$$\left[\alpha\left(k\beta-\frac{2u}{\alpha^2}\right)\frac{1}{\lambda_1(u)}e^{-\lambda_1(u)x_1}+\alpha k\frac{1}{\lambda_2(u)}e^{-\lambda_2(u)x_2}+\alpha kx_2\right]<\left[\frac{\alpha k\beta\sigma_1}{\sqrt{2\alpha}}e^{-\frac{\sqrt{2\alpha}}{\sigma_1}x_1}+\frac{2u}{\alpha}\left(x_1-\frac{u}{\alpha}\right)-c(u)\right],$$

where $\lambda_2(u)$ and $\lambda_1(u)$ are described in (3.45) and (3.34), respectively, then $J(x_1,x_2,u)<J(x_1,x_2,0)$.

Proof. Using (3.30) we observe that $J(x_1,x_2,u)<J(x_1,x_2,0)$ if and only if

$$\Psi(x_1,x_2,u)-\Psi(x_1,x_2,0)<[\Phi(x_1,0)-\Phi(x_1,u)]-\frac{c(u)}{\alpha}. \qquad (3.48)$$

Next we can use the estimate (3.46) in Proposition 3.10. Therefore, the inequality

$$\left[k\beta\left(\frac{1}{\lambda_1(u)}e^{-\lambda_1(u)x_1}-\frac{\sqrt{2\alpha\sigma_1^2}}{2\alpha}e^{-\frac{\sqrt{2\alpha}}{\sigma_1}x_1}\right)+\frac{k}{\lambda_2(u)}e^{-\lambda_2(u)x_2}+kx_2\right] < [\Phi(x_1,0)-\Phi(x_1,u)]-\frac{c(u)}{\alpha}$$

implies the inequality in (3.48). Using (3.34) and following a straightforward computation, we obtain

$$\left(k\beta - \frac{2u}{\alpha^2}\right)\frac{1}{\lambda_1(u)}e^{-\lambda_1(u)x_1} + \frac{k}{\lambda_2(u)}e^{-\lambda_2(u)x_2} + kx_2 \;<\; \frac{2u}{\alpha^2}\left(x_1 - \frac{u}{\alpha}\right) + \frac{k\beta\sigma_1}{\sqrt{2\alpha}}e^{-\frac{\sqrt{2\alpha}}{\sigma_1}x_1} - \frac{c(u)}{\alpha}.$$

This inequality is same as (3.47) and hence the result follows. ■

Remark 3.13

1. If x_2 and u remain fixed and x_1 tends to infinity then the right hand side of the inequality in (3.47) tends to infinity while the left hand side of (3.47) tends to $\frac{\alpha k}{\lambda_2(u)}e^{-\lambda_2(u)x_2} + \alpha k x_2$. Therefore, for fixed x_2 and u, large values of x_1 satisfy (3.47).

2. If $2u_0 > \alpha^2 k\beta$ and if there is a point $(\bar{x}_1, x_2)$ that satisfies

$$\frac{2u_0}{\alpha}\left(\bar{x}_1 - \frac{u_0}{\alpha}\right) - c(u_0) - \alpha k x_2 \;>\; \frac{1}{k}\left(\sqrt{\beta^2 u_0^2 + 2\alpha\sigma_2^2} - \beta u_0\right),$$

then the above inequality holds for all $x_1 \geq \bar{x}_1$. It is a straightforward matter to check that the assumption of Proposition 3.12 is true. Hence $J(x_1, x_2, u) < J(x_1, x_2, 0)$ for all $x_1 \geq \bar{x}_1$.

Next, we introduce the region

$$\mathcal{A} = \{(x_1, x_2) : \text{ There exists } u > 0 \text{ where } (x_1, x_2, u) \text{ satisfies (3.47) }\}. \tag{3.49}$$

This set $\mathcal{A}$ is non-empty as explained in the above remark.

Theorem 3.14 *Let the initial data (x_1, x_2) of (3.19) belong to the set $\mathcal{A}$ in (3.49). Then there is an optimal control $u_d^* > 0$ such that $Q(x_1, x_2) = J(x_1, x_2, u^*)$, where Q is the value function defined in (3.33).*

Proof. We obtain $J(x_1, x_2, u) < J(x_1, x_2, 0)$ for some $u > 0$ by using Proposition 3.12. On the other hand by (3.30), $J(x_1, x_2, u) > \frac{c(u)}{\alpha}$ for all $u \geq 0$. Since $c(\cdot)$ is strictly increasing and $\lim_{u\to+\infty} c(u) = +\infty$, there exists $u_2 > 0$ such that $\frac{c(u)}{\alpha} > J(x_1, x_2, 0)$ for all $u > u_2$. Consequently, $\inf_{u\geq 0} J(x_1, x_2, u) = \inf_{0<u<u_2} J(x_1, x_2, u)$. By Proposition 3.6, $J(x_1, x_2, \cdot)$ is continuous in u variable. Therefore, there exists a u_d^* such that $0 < u_d^* < u_2$ which satisfies $J(x_1, x_2, u_d^*) =$

$\inf_{u\geq 0} J(x_1, x_2, u)$. ■

4 Asymptotic Optimality

Our objective in this section is to use the optimal controls derived for the BCPs in the previous section for the construction of asymptotically optimal controls for the queueing control problem (as described in Definition 2.6). This construction is used to prove the main theorems of the article, Theorems 2.7 and 2.8. Throughout this section, ρ^* and $\bar{\mu}^*$ (see Assumption 2.4) are fixed and for simplicity of notation, we will denote $\bar{\mu}^*$ by $\bar{\mu}$ and omit ρ^* from all notations, i.e. denote $\lambda_N^n(\rho^*), \bar{\lambda}_N(\rho^*), \lambda_R^n(\rho^*), \bar{\lambda}_R(\rho^*)$ by $\lambda_N^n, \bar{\lambda}_N, \lambda_R^n, \bar{\lambda}_R$ respectively.

In this section, we will use the standard notation $\mathcal{D}([0,\infty), I\!R)$ for the set of all right continuous functions from $[0,\infty)$ to $I\!R$ with left limits. All the processes are defined on $\mathcal{D}([0,\infty), I\!R)$ unless specified otherwise. $e \in \mathcal{D}([0,\infty), I\!R)$ will denote the identity function, i.e. $e(t) = t$ for all $t \geq 0$. The convergence in distribution of a sequence of processes $\Phi_n(\cdot)$ to $\Phi(\cdot)$ will be denoted as $\Phi_n \Rightarrow \Phi$ or by $\Phi_n(\cdot) \Rightarrow \Phi(\cdot)$. When $\sup\limits_{0\leq s\leq t} |\Phi_n(s) - \Phi(s)| \to 0$ as $n \to \infty$, for all $t \geq 0$, we will write that $\Phi_n \to \Phi$ "uniformly on compact sets", or "uniformly on compacts".

4.1 Scaled processes and a Skorohod map

We have defined two types of scalings for the various processes in (2.10) above. Here we obtain convenient representations for the rescaled processes that are relevant for our analysis. Recall the definition of u^n in (2.11). We define another similar quantity $\tilde{u}^n$ below. Note that by Assumption 2.1, these two quantities are asymptotically equivalent: There exists $u \geq 0$ such that

$$u^n = \sqrt{n}\left(\mu^n - \frac{\lambda_R^n}{\beta}\right) \to u, \quad \tilde{u}^n = \sqrt{n}\left(\mu^n - \lambda_N^n\right) \to u, \text{ as } n \to \infty. \tag{4.50}$$

Next, we introduce the martingales which are related to the Poisson processes in the heavy-traffic analysis of the queueing control problem.

$$
\begin{aligned}
\hat{M}_1^n(t) &= \frac{1}{\sqrt{n}}\left[N_1^n(n\lambda_N^n t) - n\lambda_N^n t\right], \\
\hat{M}_2^n(t) &= \frac{1}{\sqrt{n}}\left[N_2^n\left(n\int_0^t \mu^n\, 1_{\{\hat{X}_1^n(s)>0\}}ds\right) - n\int_0^t \mu^n\, 1_{\{\hat{X}_1^n(s)>0\}}ds\right], \\
\hat{M}_3^n(t) &= \frac{1}{\sqrt{n}}\left[\Phi^n\left(N_2^n\left(n\int_0^t \mu^n\, 1_{\{\hat{X}_1^n(s)>0\}}ds\right)\right) - n\beta\int_0^t \mu^n\, 1_{\{\hat{X}_1^n(s)>0\}}ds\right], \\
\hat{M}_4^n(t) &= \frac{1}{\sqrt{n}}\left[N_3^n\left(n\int_0^t \lambda_R^n\, 1_{\{\hat{X}_2^n(s)>0\}}ds\right) - n\int_0^t \lambda_R^n\, 1_{\{\hat{X}_2^n(s)>0\}}ds\right], \qquad (4.51)\\
\hat{W}_1^n(t) &= \hat{M}_1^n(t) - \hat{M}_2^n(t), \\
\hat{W}_2^n(t) &= \hat{M}_3^n(t) - \hat{M}_4^n(t), \qquad (4.52)
\end{aligned}
$$

for all $t \geq 0, n = 1, 2, \ldots$. From the definition of the processes in (2.4)-(2.6) and the diffusion scaled processes in (2.10) (with a simple change of variable formula $\int_0^{nt} g(s)ds = n\int_0^t g(ns)ds$) we can write the scaled state processes as

$$
\begin{aligned}
\hat{X}_1^n(t) &= \hat{x}_1^n + \frac{1}{\sqrt{n}}N_1^n(n\lambda_N^n t) - \frac{1}{\sqrt{n}}N_2^n\left(n\int_0^t \mu^n\, 1_{\{\hat{X}_1^n(s)>0\}}ds\right) \\
&= \hat{x}_1^n - \tilde{u}^n t + \hat{W}_1^n(t) + \hat{L}_1^n(t), \qquad (4.53)\\
\hat{X}_2^n(t) &= \hat{x}_2^n + \frac{1}{\sqrt{n}}\Phi^n\left(N_2^n\left(n\int_0^t \mu^n\, 1_{\{\hat{X}_1^n(s)>0\}}ds\right)\right) - \frac{1}{\sqrt{n}}N_3^n\left(n\int_0^t \lambda_R^n\, 1_{\{\hat{X}_2^n(s)>0\}}ds\right) \\
&= \hat{x}_2^n + \beta u^n t + \hat{W}_2^n(t) - \beta\hat{L}_1^n(t) + \hat{L}_2^n(t), \qquad (4.54)\\
\text{where} \quad & \hat{L}_1^n(t) = \mu^n\sqrt{n}\int_0^t 1_{\{\hat{X}_1^n(s)=0\}}ds, \;\; \hat{L}_2^n(t) = \lambda_R^n\sqrt{n}\int_0^t 1_{\{\hat{X}_2^n(s)=0\}}ds, \qquad (4.55)
\end{aligned}
$$

for all $n = 1, 2, \ldots, t \geq 0$.

The proof of asymptotic optimality uses the following maps and their properties.

Lemma 4.1 (A two-dimensional Skorohod map) *Let $u_1, u_2, \beta \geq 0$ and let $u = (u_1, u_2)$. For each $x_1, x_2 \geq 0$ and and $w = (w_1, w_2) \in \mathcal{D}([0,\infty), I\!R) \times \mathcal{D}([0,\infty), I\!R)$ with $w_i(0) \geq 0, i = 1, 2$, there exist unique $q_i, \ell_i \in \mathcal{D}([0,\infty), I\!R), i = 1, 2$, satisfying the following properties:*

(i) $q_1(t) = x_1 - u_1 t + w_1(t) + \ell_1(t) \geq 0, \;\; \forall t \geq 0,$

(ii) $q_2(t) = x_2 + \beta u_2 t + w_2(t) - \beta\ell_1(t) + \ell_2(t) \geq 0, \ \ \forall t \geq 0,$

(iii) $\ell_i(\cdot)$ *is nondecreasing,* $\ell_i(0) = 0$ *and* $\int_0^\infty q_i(t)d\ell_i(t) = 0$ *for* $i = 1, 2$.

Define the following maps $\Gamma_i^u, \hat{\Gamma}_i^u, i = 1, 2$ *as follows: for a given* w *as above, let* $\Gamma_i^u(w) = q_i, \hat{\Gamma}_i^u(w) = \ell_i, \ \ i = 1, 2$. *We will denote the map* $(\Gamma_i^u(\cdot), \hat{\Gamma}_i^u(\cdot) : i = 1, 2)$ *as the Skorohod map relevant for this problem.*

Proof of the existence of the above map is straightforward. For $x \in \mathcal{D}([0, \infty), I\!R)$ with $x(0) \geq 0$, define the following maps:

$$\phi(x)(t) = x(t) + \psi(x)(t), \quad \text{where } \psi(x)(t) = -\inf_{0 \leq s \leq t} \min\{x(s), 0\}, \quad \text{for } t \geq 0. \tag{4.56}$$

The above maps are called one-dimensional Skorohod maps in $[0, \infty)$ (see [23, 31]). Using the above maps it is easy to verify that if $w_i \in \mathcal{D}([0, \infty), I\!R), i = 1, 2$, the following representations hold for the Skorohod maps defined in Lemma 4.1.

$$\begin{aligned} \Gamma_1^u(w) &= \phi(x_1 - ue + w_1), \ \ \hat{\Gamma}_1^u(w) = \psi(x_1 - ue + w_1), \\ \Gamma_2^u(w) &= \phi\left(x_2 + \beta ue + w_2 - \beta\hat{\Gamma}_1^u(w)\right), \ \ \hat{\Gamma}_2^u(w) = \psi\left(x_2 + \beta ue + w_2 - \beta\hat{\Gamma}_1^u(w)\right). \end{aligned} \tag{4.57}$$

It is well known that the maps ϕ and ψ are both Lipschitz continuous maps in the uniform topology (see [23] for example). More precisely, for $x, x' \in \mathcal{D}([0, \infty), I\!R)$ and $||x||_T \doteq \sup_{0 \leq s \leq T} |x(s)|$, we have

$$||\phi(x) - \phi(x')||_T \leq C_0||x - x'||_T, \ \ ||\psi(x) - \psi(x')||_T \leq C_0||x - x'||_T, \tag{4.58}$$

for some $C_0 > 0$. Using the representations in (4.57) above, one can verify that the Skorohod maps defined in Lemma 4.1 are continuous functions in the metric of uniform convergence on compact sets in the following sense: For all $T > 0$

$$\lim_{n\to\infty} |u_i^n - u_i| = 0, \ \lim_{n\to\infty} ||w_i^n - w_i||_T = 0, \ i = 1, 2$$

implies

$$\lim_{n\to\infty} ||\Gamma_i^{u^n}(w_1^n, w_2^n) - \Gamma_i^u(w_1, w_2)||_T = 0, \ \ \lim_{n\to\infty} ||\hat{\Gamma}_i^{u^n}(w_1^n, w_2^n) - \hat{\Gamma}_i^u(w_1, w_2)||_T = 0, \ \text{for } i = 1, 2. \tag{4.59}$$

This continuity property will be crucial for establishing some of the convergence results in the proofs below.

4.2 Weak convergence analysis and proof of Theorems 2.7 and 2.8

Note that for any admissible policy $\{\mu^n\}$, there exists $u \geq 0$, such that $\tilde{u}^n$ and u^n both converge to u as n tends to infinity. To simplify notation, we will use the following abbreviation for this section: For $\bar{u}^n = (\tilde{u}^n, u^n)$ and $\bar{u} = (u, u)$, $\Gamma_i^n \doteq \Gamma_i^{\bar{u}^n}, \hat{\Gamma}_i^n \doteq \hat{\Gamma}_i^{\bar{u}^n}, \Gamma_i \doteq \Gamma_i^{\bar{u}}, \hat{\Gamma}_i \doteq \hat{\Gamma}_i^{\bar{u}}$, for $i = 1, 2$. We start with the following lemma, which describes equivalent representations of the cost functionals in the queueing network control problems as well as the Brownian control problems in Section 3.

Lemma 4.2 *The long-run average cost functionals for the queueing network and the BCP in (2.12) and (3.20), respectively, have the following representation:*

$$\begin{aligned} \hat{I}(x_1, x_2, \{\mu^n\}) &= \liminf_{n\to\infty} \left[c\left(u^n\right) + \hat{\gamma}(\{\mu^n\}) + k \limsup_{T\to\infty} \frac{1}{T} E\left(\hat{L}_2^n(T)\right)\right], \\ &\text{where} \quad \hat{\gamma}(\{\mu^n\}) = \left(1 - \frac{\lambda_N^n}{\mu^n}\right) \sum_{i=0}^{\infty} h\left(\frac{i}{\sqrt{n}}\right)\left(\frac{\lambda_N^n}{\mu^n}\right)^i, \end{aligned} \tag{4.60}$$

$$I(x_1, x_2, u) = c(u) + \gamma(u), \quad \textit{where} \quad \gamma(u) = \frac{2u}{\sigma_1^2} \int_0^\infty h(x) e^{-\frac{2ux}{\sigma_1^2}} dx. \tag{4.61}$$

The infinite horizon discounted cost functionals for the queueing network and the BCP in (2.13) and (3.29), respectively, have the following representation:

$$\hat{J}(x_1, x_2, \rho, \{\mu^n\}) = \liminf_{n\to\infty} \frac{c\left(u^n\right)}{\alpha} + E \int_0^\infty \alpha e^{-\alpha t} \left[\int_0^t \left(\hat{X}_1^n(s)\right)^2 ds + k \, \hat{L}_2^n(t)\right] dt, \tag{4.62}$$

$$J(x_1, x_2, u) = \frac{c(u)}{\alpha} + E \int_0^\infty \alpha e^{-\alpha t} \left[\int_0^t \left(X_1(s)\right)^2 ds + k \, L_2(t)\right] dt. \tag{4.63}$$

Proof. To verify (4.60), first note that for each fixed $n \geq 1$, $\hat{X}_1^n$ is a jump-Markov process with state-space $\mathcal{L}^n \doteq \left\{ \frac{j}{\sqrt{n}} : j = 0, 1, 2, \dots \right\}$, and jump rates given by

$$\mathcal{Q}^n(i,j) = \begin{cases} n\lambda_N^n, & \text{if } j = i + \frac{1}{\sqrt{n}}, i \in \mathcal{L}^n \\ n\mu^n, & \text{if } j = i - \frac{1}{\sqrt{n}}, i \in \mathcal{L}^n \setminus \{0\} \\ 0, & \text{otherwise.} \end{cases}$$

Straightforward calculations (solving the balance equations) yields that the invariant distribution for $\hat{X}_1^n$ is that of a random variable X_∞^n, where $\sqrt{n}X_\infty^n$ follows a *Geometric* distribution with parameter $a_n = 1 - (\lambda_N^n / \mu^n)$. Therefore, it follows that

$$\lim_{T \to \infty} \left[\frac{1}{T} E \int_0^T h(\hat{X}_1^n(s)) ds \right] = E\left[h(X_\infty^n) \right].$$

Note that by the assumptions on $h(\cdot)$, and distribution of X_∞, the expectation on the right side is finite. This proves that the representation in (4.60) is accurate. The representation in (4.61) follows from Proposition 3.2 and Lemma 3.3. The proof of the representations of the discounted cost functionals in (4.62)-(4.63) are standard and similar to that of Lemma 4.4 of [12]. ■

Proposition 4.3 *The processes $\hat{X}_1^n$ and $\hat{X}_2^n$ in (4.53)-(4.54) satisfy*

$$(\hat{X}_1^n, \hat{X}_2^n, \hat{L}_1^n, \hat{L}_2^n) = \left(\Gamma_1^n(\hat{W}^n), \Gamma_2^n(\hat{W}^n), \hat{\Gamma}_1^n(\hat{W}^n), \hat{\Gamma}_2^n(\hat{W}^n) \right). \tag{4.64}$$

For the processes $\hat{W}_1^n$ and $\hat{W}_2^n$ defined in (4.52), the following convergence holds:

$$\hat{W}^n \equiv (\hat{W}_1^n, \hat{W}_2^n) \Rightarrow (W_1, W_2) \equiv W, \tag{4.65}$$

where W is a two-dimensional Brownian motion as described in (3.19). Also, the following holds.

$$\begin{aligned} (\hat{X}_1^n, \hat{X}_2^n, \hat{L}_1^n, \hat{L}_2^n) \quad &\Rightarrow \quad (X_1, X_2, L_1, L_2) \text{ and } n \to \infty, \\ \text{where } (X_1, X_2, L_1, L_2) &\doteq \left(\Gamma_1(W), \Gamma_2(W), \hat{\Gamma}_1(W), \hat{\Gamma}_2(W) \right), \end{aligned} \tag{4.66}$$

and $(W_1, W_2, X_1, X_2, L_1, L_2)$ *satisfies all the conditions on the processes involved in defining the BCPs in (3.19).*

The proof of the above proposition is somewhat standard in the heavy traffic literature. We skip the proof here, and present one in the Appendix. The following basic lemma will be used in our proof of the main result. A proof of this lemma can be found in the Appendix as well.

Lemma 4.4 *Let* $\{a_n\}$ *be a sequence such that* $a_n \to a$ *as* $n \to \infty$ *and* $h(\cdot)$ *be the cost function used in our analysis. Then the following holds*

$$a_n \sum_{k=0}^{\infty} h\left(\frac{k}{\sqrt{n}}\right)\left(1-\frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \quad \to \quad a\int_0^{\infty} h(x)e^{-ax}dx, \text{ as } n\to\infty. \tag{4.67}$$

We will use the following moment estimates in our analysis to establish the convergence of the cost functionals.

Proposition 4.5 *The following estimates hold: There exist constants* $C_1, C_2 > 0$ *such that for all* $n \geq 1$ *and* $t \geq 0$

$$E\left[\sup_{0\le s\le t} \mid \hat{X}_1^n(s) \mid^4\right] \;\le\; C_1\left(1+t^2+t^4\right) \;\text{ and} \tag{4.68}$$

$$E\left[\left(\hat{L}_2^n(t)\right)^2\right] \;\le\; C_2\left(1+t+|u^n-\tilde{u}^n|^2t^2\right), \tag{4.69}$$

where u *and* u^n *are as described in (4.50).*

Proof. From the representation of $\hat{X}_1^n$ in (4.66) and (4.57)-(4.58), we have

$$\begin{aligned} E\left[\sup_{0\le s\le t} \mid \hat{X}_1^n(s) \mid^4\right] &\le C_0^4\, E\left[\sup_{0\le s\le t} \mid x_1^n - \tilde{u}^n s + \hat{W}_1^n(s) \mid^4\right] \\ &\le C\, E\left[(x_1^n)^4 + (\tilde{u}^n)^4 t^4 + E\left(\sup_{0\le s\le t} \mid \hat{W}_1^n(s) \mid^4\right)\right], \end{aligned} \tag{4.70}$$

for some constant $C > 0$, independent of n and t. Since $\{x_1^n\}$ and $\{\tilde{u}^n\}$ are both convergent sequences, by Doob's inequality for the martingale $\hat{W}_1^n(\cdot)$, and using the fact that $E(\hat{W}_1^n(t))^4 \le C't^2$ (for some $C' > 0$) we have the proof of the first estimate (4.68) of the proposition.

For the second estimate, note that from (4.53), we have

$$\begin{aligned} E\left[\sup_{0\le s\le t} \mid u^n s - \hat{L}_1^n(s) \mid^2\right] &\le C\left[|u^n - \tilde{u}^n|^2 t^2 + E\left(\sup_{0\le s\le t} \mid \tilde{u}^n s - \hat{L}_1^n(s) \mid^2\right)\right] \\ &\le C\left[|u^n - \tilde{u}^n|^2 t^2 + (x_1^n)^2 + E\left(\sup_{0\le s\le t} \mid \hat{W}_1^n(s) \mid^2\right) + E\left(\sup_{0\le s\le t} \mid \hat{X}_1^n(s) \mid^2\right)\right] \\ &\le C\left(|u^n - \tilde{u}^n|^2 t^2 + 1 + t + t^2\right), \end{aligned} \tag{4.71}$$

where the last estimate follows using (4.68) and arguments similar to those used in obtaining (4.70). Here, $C > 0$ represents a generic constant independent of n and t and the value of this constant varies from line to line of (4.71). Now, from the representation of $\hat{L}_1^n$ in (4.53), we have

$$E\left[\left(\hat{L}_2^n(t)\right)^2\right] \le C_0^2\left[(x_2^n)^2 + E\left(\sup_{0\le s\le t} \mid \hat{W}_2^n(s) \mid^2\right) + \beta E\left(\sup_{0\le s\le t} \mid u^n s - \hat{L}_1^n(s) \mid^2\right)\right].$$

The second estimate now follows from (4.71), the Doob's inequality for the martingale $\hat{W}_2^n(\cdot)$, and the fact that $E(\hat{W}_2^n(t))^2 \le C't$, for some $C' > 0$. ■

Now, using the results above, we prove the main theorems of the paper, viz. Theorems 2.7 and 2.8, regarding optimal controls for the queueing network control problem.

Proof of Theorem 2.7: First, we prove the asymptotic analysis of our proposed policy for the ergodic cost problem. Since for any admissible policy $\{\mu^n\}$, the corresponding $\{u^n\}$ converges to some $u \ge 0$, we have from the continuity of $c(\cdot)$ that

$$c(u^n) \to c(u), \text{ as } n \to \infty. \tag{4.72}$$

Also, note that for such policies, $a_n \doteq \sqrt{n}(1 - \frac{\lambda_N^n}{\mu^n}) = (\tilde{u}^n/\mu^n)$ converges to $a \doteq \frac{u}{\bar{\mu}} = \frac{2u}{\sigma_1^2}$, since $\sigma_1^2 = \bar{\lambda}_N + \bar{\mu} = 2\bar{\mu}$ by Assumption 2.4. Therefore, by Lemma 4.4, we have

$$\hat{\gamma}^n(\{\mu^n\}) \to \gamma(u), \text{ as } n \to \infty, \tag{4.73}$$

where $\hat{\gamma}^n$ and γ are as described in (4.60) and (4.61). By Proposition 4.5, we have that

$$\liminf_{n\to\infty}\left[\limsup_{T\to\infty}\frac{1}{T}E\hat{L}_2^n(T)\right] \;\leq\; \liminf_{n\to\infty}\left[\limsup_{T\to\infty}\frac{1}{T}\sqrt{C_2(1+T+|u^n-\tilde{u}^n|^2T^2)}\right] \;=\; 0. \qquad (4.74)$$

Hence by Lemma 4.2, we have that for any admissible control policy $\{\mu^n\}$, with u^n converging to some $u \geq 0$,

$$\hat{I}(x_1, x_2, \{\mu^n\}) \;=\; I(x_1, x_2, u). \qquad (4.75)$$

Note that from the construction of our proposed policy $\{\mu_a^{n,*}\}$, we have that the corresponding $\{u^{n,*}\}$ converges to $u_a^* \geq 0$, where u_a^* is as in Theorem 3.4. Hence we have that from Theorem 3.4 and (4.75) that for any admissible policy $\{\mu^n\}$

$$\hat{I}(x_1, x_2, \{\mu^n\}) \;=\; I(x_1, x_2, u) \;\geq\; I(x_1, x_2, u_a^*) = \hat{I}(x_1, x_2, \{\mu_a^{n,*}\}). \qquad (4.76)$$

This proves the asymptotic optimality of our proposed policy for the queueing network problem with long-run average cost. ∎

Proof of Theorem 2.8: Now we prove the optimality result for the infinite horizon discounted cost problem. Note that for any admissible control policy $\{\mu^n\}$ with the corresponding $\{u^n\}$ converging to some $u \geq 0$, we have from Proposition 4.3 that

$$\int_0^{\cdot}(\hat{X}_1^n(s))^2 ds \;\Rightarrow\; \int_0^{\cdot}(X_1(s))^2 ds, \text{ as } n\to\infty. \qquad (4.77)$$

This follows from the fact that $\hat{X}_1^n \Rightarrow X_1$ and the function $\int_0^{\cdot} x(s)^2 ds$ is a continuous map on $\mathcal{D}([0,\infty), I\!R)$ with respect to the uniform metric on compacts, and hence continuous mapping theorem applies. Combining the last part of Proposition 4.3 and (4.77), we have for all fixed $t \geq 0$

$$\left[\int_0^t \left(\hat{X}_1^n(s)\right)^2 ds + k\,\hat{L}_2^n(t)\right] \;\Rightarrow\; \left[\int_0^t (X_1(s))^2\, ds + k\, L_2(t)\right], \text{ as } n\to\infty. \qquad (4.78)$$

Observe that from Proposition 4.5, we have for each fixed $t \geq 0$,

$$E\left[\int_0^t \left(\hat{X}_1^n(s)\right)^2 ds + k\,\hat{L}_2^n(t)\right]^2 \;\leq\; C_3[1+t^2+t^4], \text{ for all } n \geq 1, \tag{4.79}$$

where $C_3 > 0$ is a constant independent of n. From (4.78) and (4.79) we get that for each fixed $t \geq 0$,

$$E\left[\int_0^t \left(\hat{X}_1^n(s)\right)^2 ds + k\,\hat{L}_2^n(t)\right] \;\rightarrow\; E\left[\int_0^t (X_1(s))^2\, ds + k\,L_2(t)\right], \text{ as } n \rightarrow \infty. \tag{4.80}$$

Now from (4.79), it is easy to verify that

$$\int_0^\infty \alpha e^{-\alpha t} E\left[\int_0^t \left(\hat{X}_1^n(s)\right)^2 ds + k\,\hat{L}_2^n(t)\right]^2 dt \;\leq C_4 \text{ for all } n \geq 1, \tag{4.81}$$

where $C_4 > 0$ is a constant independent of n and t. This bound together with the convergence in (4.80) implies that as $n \rightarrow \infty$,

$$\int_0^\infty \alpha e^{-\alpha t} E\left[\int_0^t \left(\hat{X}_1^n(s)\right)^2 ds + k\,\hat{L}_2^n(t)\right] dt \;\rightarrow\; \int_0^\infty \alpha e^{-\alpha t} E\left[\int_0^t (X_1(s))^2\, ds + k\,L_2(t)\right] dt. \tag{4.82}$$

Hence by Lemma 4.2, we have that for any admissible control policy $\{\mu^n\}$, with u^n converging to some $u \geq 0$,

$$\hat{J}(x_1, x_2, \{\mu^n\}) \;=\; J(x_1, x_2, u). \tag{4.83}$$

Using the same arguments as in obtaining (4.75), we get

$$\hat{J}(x_1, x_2, \{\mu^n\}) \;=\; J(x_1, x_2, u) \;\geq\; J(x_1, x_2, u_d^*) = \hat{J}(x_1, x_2, \{\mu_d^{n,*}\}), \tag{4.84}$$

where $u_d^* \geq 0$ is the optimal drift for the BCP as given in Theorem 3.14. ■

5 Comparative Statics and Numerical Analysis

In this section, we examine the sensitivity of the asymptotically optimal policy to changes in the problem parameters. In our stylized model, the consumer willingness-to-pay for refurbished items, δ, and the return fraction, β, are treated as exogenous parameters, but in reality they may be influenced by the producer. For instance, marketing efforts or warranties could increase δ while generous return policies would increase β. On the other hand, costs of backorders for new products and lost sales of refurbished products are difficult to estimate. We investigate the impact of β and δ on the optimal production rate. We focus on the case where the backorder and service costs are polynomials and vary the constant cost, k, per lost sale of refurbished product.

From the expressions given in Remark 2.5, it is easy to verify that under the heavy traffic conditions,

$$\frac{\partial \bar{\lambda}_N(\rho^*)}{\partial \delta} < 0, \ \frac{\partial \bar{\lambda}_R(\rho^*)}{\partial \delta} < 0, \ \frac{\partial \bar{\lambda}_N(\rho^*)}{\partial \beta} < 0, \ \text{ and } \frac{\partial \bar{\lambda}_R(\rho^*)}{\partial \beta} > 0.$$

When minimizing average cost, Theorem 3.4 implies that only the costs associated with the "forward" portion of the closed loop supply chain (i.e., those that deal with new products) influence the optimal policy. The usual tradeoff between backorder and production costs exists, and the decrease with both δ and β in the rate of orders for new products suggests that the optimal control u_a^* should decrease with respect to both parameters. Indeed, for the case where $c(x) = x^m$ and $h(x) = x^q$, by substituting $\sigma_1 = \sqrt{2\bar{\lambda}_N(\rho^*)}$ in (3.28), we find

$$\frac{\partial u_a^*}{\partial \beta} = -\frac{q\delta}{(m+q)(1+\beta\delta)} \left[\frac{qq!}{m} \left(\frac{1-p_N}{1+\beta\delta} \right)^q \right]^{\frac{1}{m+q}} < 0$$

and

$$\frac{\partial u_a^*}{\partial \delta} = \frac{\beta}{\delta} \frac{\partial u_a^*}{\partial \beta} < 0.$$

In the infinite horizon discounted cost case, the situation is more complicated because the lost sales cost persists in the diffusion limit and the optimal control also depends on the initial length of the queue for new products, x_1. (Here we assume $x_2 = 0$ because if time zero represents the start of production, there would be no initial inventory of refurbished products.) The effect of δ on the optimal control is predictable because the demands for new and refurbished products both decrease with δ under the heavy traffic conditions; therefore, we expect (and have verified numerically) that for given x_1 and k, $u_d^* \equiv \arg\min_{u_d} J(x_1, 0, u_d)$ decreases with δ. However, because the relationship of u_d^* with β is not so clear either mathematically or intuitively, we resort to numerical analysis. Analytical results from the Brownian control problem guarantee continuity of the cost function with respect to u and existence of optimal u_d^*, and also provide an exact closed form expression for one component of the cost. Simulation is required to approximate u_d^* for any particular set of parameters.

Let $c(u) = u^2$, $h(x) = x^2$ and consider $p_N = 0.9$, $\alpha = 0.1$, and $\delta = 0.65$ (Hauser and Lund [20] report that remanufactured products are typically sold at 45% to 65% of the price of comparable new products). To numerically optimize the infinite horizon expected discounted cost, we generate two independent standard Brownian motion processes $B_1(t)$ and $B_2(t)$ for $t = 0, 0.1, 0.2, ..., 10^5$. Let $W_1(t) = B_1(t)$ and $W_2(t) = -rB_1(t) + \sqrt{1-r^2}B_2(t)$, where $r = \frac{\bar{\mu}\beta}{\sqrt{(\bar{\lambda}_N+\bar{\mu})(\bar{\lambda}_R+\bar{\mu}\beta(1-\beta))}}$. Computing X_1, L_1, L_2 from (W_1, W_2) using (4.66) of Proposition 4.3, we approximate $J(x_1, 0, u_d)$ from the sample mean of 1000 realizations, and obtain the approximate optimal $u \cong u_d^*$. The sample size is sufficiently large to make the standard error of the estimate of the cost less than or equal to 1% of its value. Figure 3 shows a plot of the (estimated) $J(5, 0, u_d)$ for $\beta = 0.1, \delta = 0.65, k = 5$ as a function of u_d. While we have been able to prove only that $u_d^* > 0$ exists, the discounted cost function appears to be convex. Figure 4 plots the estimated u_d^* against β for various k given $x_1 = 0$ and $x_1 = 5$; the optimal u_a^* for the ergodic cost $F(u_a)$ is also shown for reference. Observe that the optimal control increases with k, as expected, because increasing lost sales costs prompt faster production for a fixed return fraction to increase the supply of returned products to refurbish. The optimal control u_d^* does not show monotonicity with respect to β for either $x_1 = 0$ or $x_1 = 5$, which highlights the complexity of managing the closed loop supply chain under uncertainty.

We also compared results from the average and discounted cost cases numerically as $\alpha \to 0$ for the same parameter values along with $x_1 = 5$, $x_0 = 0$, $\beta = 0.1$ and $k = 1$. Table 1 suggests that as the discount parameter α decreases to zero, the optimal value functions are asymptotically related by $\alpha J(5, 0, u_d^*(\alpha)) \to F(u_a^*)$ and also $u_d^*(\alpha) \to u_a^*$ holds. These results suggest that an Abelian limit relationship for the value functions of the infinite horizon discounted cost problem and long-run average cost problem could be extended to this setting. For one-dimensional diffusion models, such Abelian limit theorems were established in [35].

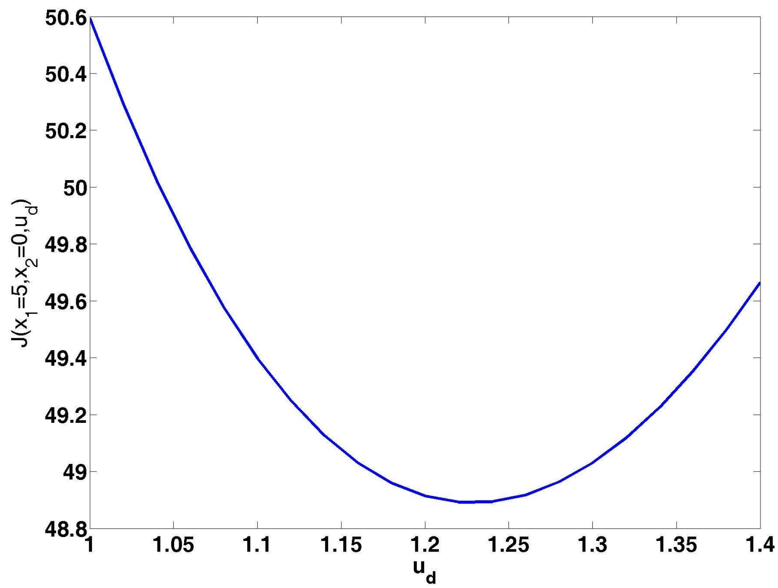

Figure 3: Discounted cost function $J(x_1 = 5, x_2 = 0, u_d)$ for $\beta = 0.1, \delta = 0.65, k = 5$.

Table 1: Convergence of $\alpha J(5, 0, u_d^*(\alpha))$ and $u_d^*(\alpha)$ to $F(u_a^*) = 0.2656$ and $u_a^* = 0.3644$.

α	$\alpha J(5, 0, u_d^*(\alpha))$	$u_d^*(\alpha)$
0.1	4.8537	1.2272
0.01	1.1483	0.6306
0.001	0.3612	0.3975
0.0001	0.2441	0.3558

6 Discussion

In this paper, we have examined a simple model of a closed loop supply chain and performed what to our knowledge is the first heavy traffic analysis of such a system. By solving a static planning problem to maximize profit, we derived the price ratio between new and remanufactured products that would achieve heavy traffic. Then we established the existence of asymptotically optimal controls for both average and discounted costs in the diffusion limit, considering costs of backorders for new products, lost sales of refurbished products, and manufacturing new products. From these controls, for each cost functional we derived an asymptotically optimal sequence of service rates for the queuing control problem as the system approaches heavy traffic. An important insight resulting from the mathematical analysis is that the control that minimizes long-run average cost per unit time is not influenced by the cost component from refurbished product lost sales. We showed analytically that the limiting average cost optimal production rate decreases with both the

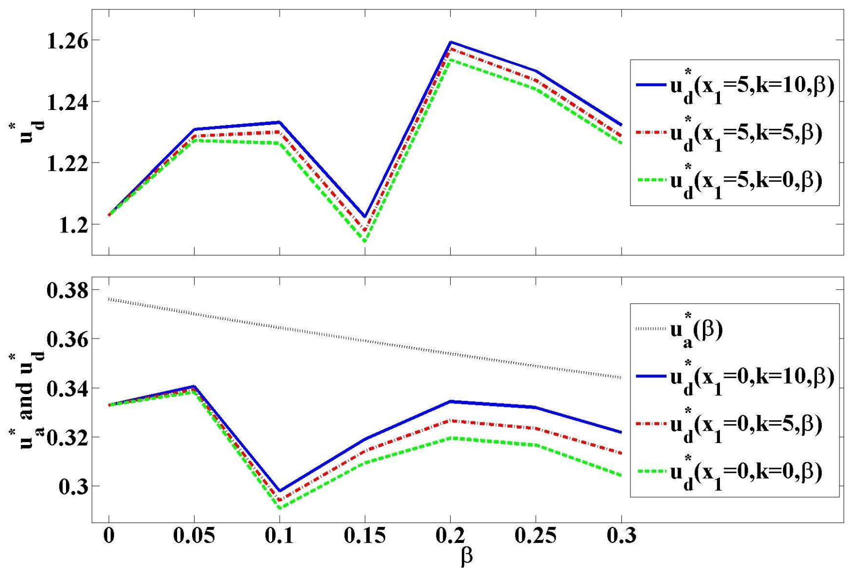

Figure 4: Control that minimizes infinite horizon discounted cost for $x_1 = 0$ and $x_1 = 5$ respectively.

product return rate and the relative amount consumers are willing to pay for refurbished products. By numerical analysis we found that the willingness-to-pay parameter has a similar effect on the limiting discounted cost optimal production rate but the effect of the return rate varies with initial conditions and the magnitude of the refurbished product lost sales cost.

Pre-Limit Analysis: In the average-cost case, the classical steady-state analysis of identical systems can be obtained in special cases. Here we compare that approach with our analysis for a sequence of systems in steady-state and approaching the heavy traffic limit.

The model satisfies the assumptions of a Jackson open queueing network. Therefore, in the sequence of networks approaching heavy traffic, if for each n we have $\lambda_N^n(\rho^*) < \mu^n$ and $\beta\lambda_N^n(\rho^*) < \lambda_R^n(\rho^*)$, then a steady state exists and arrivals of refurbished products to the second queue follow a Poisson process with rate $\beta\lambda_N^n(\rho^*)$. For example, suppose $\lambda_R^n(\rho) = \bar{\lambda}_R(\rho) + \theta_2/\sqrt{n}$ and $\lambda_N^n = \bar{\lambda}_N(\rho) + \beta\theta_2/\sqrt{n} - 1/f(n)$, where $\theta_2 > 0$ and $0 < f(n)\ \omega(\sqrt{(n)})$, i.e., $\lim_{n\to\infty} \sqrt{n}/f(n) \to 0$. Then Assumption 2.1 is satisfied and $\lambda_N^n(\rho^*) < \lambda_R^n(\rho^*)/\beta$, so that a steady state exists for each n at the optimal price provided that $\mu^n > \lambda_N^n(\rho^*)$. For the remainder of this section, we assume $\rho = \rho^*$ and suppress it in the notation.

Arrivals of refurbished products to the second queue follow a Poisson process with rate $\beta\lambda_N^n$.

The long run average cost in the n^{th} system that corresponds to equation (2.12) is:

$$\begin{aligned} I(x_1, x_2, u^n) &= c(u^n) + \limsup_{T\to\infty} E\frac{1}{T}\int_0^T h\left(\frac{X_1^n(t)}{\sqrt{n}}\right) dt + \limsup_{T\to\infty} k\lambda_R^n \sqrt{n} E\frac{1}{T}\int_0^T 1_{\{X_2^n(s)=0\}} ds \\ &= c(u^n) + Eh\left(\frac{X_1^n(\infty)}{\sqrt{n}}\right) + k \limsup_{T\to\infty} \frac{E\left(\hat{L}_2^n(T)\right)}{T}. \end{aligned}$$

Note that, under Assumptions 2.1 and 2.4, the third term vanishes as $n \to \infty$, as shown in (4.74). The second term can be evaluated as:

$$\sum_{j=0}^{\infty} h(j/\sqrt{n}) \left(\frac{\lambda_N^n}{\mu^n}\right)^j \left(1 - \frac{\lambda_N^n}{\mu^n}\right).$$

In particular, if $h(x) = x^2$, this backorder cost equals:

$$\frac{\lambda_N^n(\mu^n + \lambda_N^n)}{n(\mu^n - \lambda_N^n)^2} = \frac{\lambda_N^n \left(\frac{u^n}{\sqrt{n}} + \frac{\lambda_R^n}{\beta} + \lambda_N^n\right)}{\left(u^n + \frac{\lambda_R^n\sqrt{n}}{\beta} - \lambda_N^n\sqrt{n}\right)^2} \to \frac{2(\bar{\lambda}_N)^2}{u^2}.$$

This expression agrees with the result of Theorem 3.4. Note that for general functions $h(\cdot)$, it is not feasible to evaluate the expected backorder cost in closed form. The scaling of the queue length by $\sqrt{n}$ is a reminder that as the sequence of networks approaches heavy traffic, the average length of the queue of waiting customers will increase. However, if the steady-state backorder cost can be approximated, this expression suggests that in practice, the system could be designed by choosing n to achieve a tolerable magnitude for $E[h(X_1^n(\infty))]$ and then using the result of Theorem 2.7 to set find a near-optimal value for μ^n, that achieves an appropriate tradeoff between service cost and backorder cost.

For the infinite-horizon discounted cost, in principle the backorder cost portion could be evaluated for the n^{th} system in steady state using known results on the transient behavior of the M/M/1 queue (see [13], p.98). However, the lack of an exact representation for the arrival process to the second queue in the transient phase precludes evaluation of the lost sales portion of the discounted cost function. The product form of the steady-state distribution of queue lengths is based on the departure process from the first queue being a Poisson process, but this holds only in steady-state.

Extensions and Future Directions: We assume that all returned products are selected for refurbishment. This restriction can be removed in a straightforward manner, if we assume that for every purchased product, there is a probability $\beta_1 \in (0,1)$ of being returned by the customer, and out of those returned, each has a probability $\beta_2 \in (0,1)$ of being selected for refurbishment. This case is actually covered by our model with $\beta = \beta_1\beta_2$. We also assumed refurbishment is instantaneous and product returns, if they occur, do so immediately after purchase. To extend our model to incorporate refurbishment lead times, one can replace the N_2 term in (2.5) by

$$N_4\left(\int_0^t \gamma\Phi\left(N_2\left(\int_0^u \mu\, 1_{\{X_1(s)>0\}} ds\right)\right) du\right),$$

where $N_4(\cdot)$ is another unit Poisson process, independent of all the other variables and processes, and $\gamma > 0$. In this representation, γ is the "delay rate" for refurbishment under the assumption that refurbishment takes a random amount of time following an exponential distribution with rate γ. This is appropriate if most returned products require very little effort to refurbish but a few have serious defects requiring lengthy repairs. Similarly, an exponentially-distributed delay between purchase and return of a new product can be included by nesting another unit Poisson process with its corresponding delay rate. The analysis of such a model is similar to the one considered here, but for simplicity we did not consider such generalizations in this paper.

Our model does not include a variable storage cost for refurbished products. Including such a cost would require additional restrictions to control the length of the second queue. This would lead to a truly two-dimensional control problem and require a much more difficult analysis. We proved existence of an asymptotically optimal control for infinite horizon discounted cost only for the specific backorder cost function $h(x) = x^2$ but arbitrary convex functions should be considered. Finally, analytical explorations of the comparative statics for the discounted cost case as well as Abelian limits, available only numerically so far, could be attempted.

Acknowledgments: Arka Ghosh is supported by the NSF grant DMS-0608634. Ananda Weerasinghe is supported by the ARO grants W911NF0510032 and W911NF0710424. We are grateful for their support.

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7 Appendix

Proof of Proposition 3.2: Consider the function Q defined on $[0,\infty)$ by

$$Q(x) = \frac{2}{\sigma_1^2}\int_0^x e^{\frac{2u}{\sigma_1^2}r}\int_0^r (\gamma(u)-h(y))e^{-\frac{2u}{\sigma_1^2}y}dydr. \tag{7.85}$$

Then Q satisfies

$$\frac{\sigma_1^2}{2}Q'' - uQ' + h(x) = \gamma(u) \text{ for } x>0 \text{ and } Q'(0)=0. \tag{7.86}$$

Using (3.22) and (7.85), we also obtain

$$\begin{aligned} Q'(x) &= \frac{2}{\sigma_1^2}e^{\frac{2u}{\sigma_1^2}x}\int_0^x(\gamma(u)-h(y))e^{-\frac{2u}{\sigma_1^2}y}dy \\ &= \frac{2}{\sigma_1^2}e^{\frac{2u}{\sigma_1^2}x}\int_x^\infty(h(y)-\gamma(u))e^{-\frac{2u}{\sigma_1^2}y}dy. \end{aligned} \tag{7.87}$$

Therefore, $Q'(x)>0$ for all x and $Q(0)=0$, and consequently $Q(x)>0$ for all $x>0$.

Next we consider the process $X_1(t)$ in (3.19), and introduce the stopping time τ_N for each $N\geq 1$ as follows:

$$\tau_N = \begin{cases} \inf\{t\geq 0 : X_1(t)\geq N\}, & \text{if the above set is non-empty,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $a\wedge b \doteq \min\{a,b\}$. We apply Itô's lemma to $Q(X_1(t\wedge\tau_N))$ and use (7.86) to obtain

$$E[Q(X_1(t\wedge\tau_N))] + E\int_0^{t\wedge\tau_N} h(X_1(s))ds = Q(x) + \gamma(u)E[t\wedge\tau_N]. \tag{7.88}$$

Next we intend to estimate $E[Q_1(t\wedge\tau_N)]$. Using (7.87), we have

$$0 < Q'(x) \leq \left(\frac{2}{\sigma_1^2}\right)e^{\frac{2u}{\sigma_1^2}x}\int_x^\infty h(y)e^{-\frac{2u}{\sigma_1^2}y}dy. \tag{7.89}$$

But $h(\cdot)$ has polynomially-bounded growth and, therefore, for any $0<\epsilon<\frac{u}{\sigma_1^2}$, we have $0<h(y)<K_\epsilon e^{\epsilon y}$ for all $y>0$. Here, $K_\epsilon>0$ is a positive constant which may depend on $\epsilon>0$. Combining this with (7.89), we obtain $0 < Q'(x) \le \left(2/\sigma_1^2\right) \left(K_\epsilon e^{\epsilon x}/(\frac{2u}{\sigma_1^2} - \epsilon)\right)$, for all $x \ge 0$. Upon integration, we have $0 < Q(x) \le \left(2/\sigma_1^2\right) \left(K_\epsilon e^{\epsilon x}/\epsilon(\frac{2u}{\sigma_1^2} - \epsilon)\right)$ for all $x \ge 0$. Therefore,

$$E[Q(X_1(t \wedge \tau_N))] \le \tilde{K}_\epsilon E[e^{\epsilon X_1(t \wedge \tau_N)}], \tag{7.90}$$

where $\tilde{K}_\epsilon$ is a positive constant. We choose $\delta = 2\epsilon$ and apply Itô's lemma to obtain,

$$\begin{aligned} E[e^{\delta X_1(T \wedge \tau_N)}] &= e^{\delta x_1} + \delta E \int_0^{t \wedge \tau_N} \left(\frac{\sigma_1^2}{2}\delta - u\right) e^{\delta X_1(s)} ds + \delta E[L_1(t \wedge \tau_N)] \\ &\le e^{\delta x_1} + \delta E[L_1(t \wedge \tau_N)]. \end{aligned}$$

By (3.19), $E[L_1(t \wedge \tau_N)] = uE[t \wedge \tau_N] + E[X_1(t \wedge \tau_N)] - x_1$. Hence, $E[e^{\delta X_1(T \wedge \tau_N)}] \le e^{\delta x_1} + u\,\delta\,T + \delta\,E[X_1(t \wedge \tau_N)]$. Notice that $E[|X_1(T \wedge \tau_N)|] \le 1 + E[X_1(T \wedge \tau_N)^2]$ and again with the use of Itô's lemma, we obtain

$$\begin{aligned} E[|X_1(T \wedge \tau_N)|^2] &= x_1^2 + \sigma_1^2 E[t \wedge \tau_N] - 2uE \int_0^{t \wedge \tau_N} X_1(s) ds \\ &\le x_1^2 + \sigma_1^2 T. \end{aligned}$$

Consequently,

$$E[e^{\delta X_1(T \wedge \tau_N)}] \le e^{\delta x_1} + \delta x_1^2 + (u\delta + \sigma_1^2 \delta)T + \delta. \tag{7.91}$$

Since $\delta = 2\epsilon$, we have

$$E[e^{\epsilon X_1(T \wedge \tau_N)}] \le \sqrt{E[e^{\delta X_1(T \wedge \tau_N)}]} \le \sqrt{e^{\delta x_1} + \delta x_1^2 + \delta(u + \sigma_1^2)T + \delta}$$

Using (7.90), (7.91) and the above estimate, we obtain

$$E[Q(X_1(t \wedge \tau_N))] \le \tilde{K}_\epsilon \sqrt{e^{2\epsilon x_1} + 2\epsilon x_1^2 + 2\epsilon(u + \sigma_1^2)T + 2\epsilon}$$

We use this estimate together with (7.88) to obtain

$$\begin{aligned}\left|E\int_0^{T\wedge\tau_N} h(X_1(s))ds - \gamma(u)E[T\wedge\tau_N]\right| &\leq Q(x) + E[Q(X_1(t\wedge\tau_N))] \\ &\leq Q(x) + \tilde{K}_\epsilon\sqrt{e^{2\epsilon x_1} + 2\epsilon x_1^2 + 2\epsilon(u+\sigma_1^2)T + 2\epsilon}.\end{aligned}$$

Next we let τ_N increase to $+\infty$ and divide it by T to obtain $\lim_{T\to\infty}\frac{1}{T}E\int_0^T h(X_1(s))ds = \gamma(u)$. ■

Proof of Lemma 3.5: Fix $u \geq 0$, Notice that $\Phi(\cdot, u)$ given in (3.34) satisfies the differential equation

$$\frac{\sigma_1^2}{2}Y'' - uY' - \alpha Y + x^2 = 0 \text{ for } x > 0 \text{ and } Y'(0) = 0. \quad (7.92)$$

Since $u \geq 0$ is fixed, we relabel $\Phi(x, u)$ by $\Phi(x)$. We apply Itô's lemma to $\Phi(X_1(t))e^{-\alpha t}$ and using (7.92) we obtain

$$E[\Phi(X_1(T\wedge\tau_N))e^{-\alpha(T\wedge\tau_N)}] = \Phi(x_1) - E\int_0^{T\wedge\tau_N} e^{-\alpha t}X_1(t)^2dt, \quad (7.93)$$

where (τ_N) is a sequence of stopping times defined by

$$\tau_N = \begin{cases} \inf\{t \geq 0 : X_1(t) \geq N\}, & \text{if the above set is non-empty,} \\ +\infty, & \text{otherwise.}\end{cases}$$

Using (3.34), we observe that $|\Phi(x)| \leq H(1 + x^2)$, for all $x \geq 0$, where $C > 0$ is a generic constant. Therefore,

$$E\left[|\Phi(X_1(T\wedge\tau_N))|e^{-\alpha(T\wedge\tau_N)}\right] \leq H(1 + X_1(T\wedge\tau_N)^2)e^{-\alpha(T\wedge\tau_N)}. \quad (7.94)$$

To prove the assertion of the lemma, we intend to show that $\lim_{T\to\infty}\lim_{N\to\infty}E\left[|\Phi(X_1(T\wedge\tau_N))|e^{-\alpha(T\wedge\tau_N)}\right] = 0$ and use it together with (7.93). For this, first we apply Itô's lemma to $X_1(t)^2e^{-\epsilon t}$ for any fixed $\epsilon > 0$ and obtain the upper bound $E\left[X_1(T\wedge\tau_N)^2e^{-\epsilon(T\wedge\tau_N)}\right] \leq x_1^2 + \frac{\sigma_1^2}{\epsilon}$ for any $t > 0$. By letting N tend to infinity and using Fatou's lemma, we have $E\left[X_1(t)^2e^{-\epsilon t}\right] \leq x_1^2 + \frac{\sigma_1^2}{\epsilon}$ for any $t > 0$. Using this estimate together with Itô's lemma for $X_1^4(t)e^{-\epsilon t}$, we obtain

$$\begin{aligned} E\left[X_1^4(T\wedge\tau_N)e^{-\epsilon(T\wedge\tau_N)}\right] &\leq x_1^4 + 6\sigma_1^2 E\int_0^{T\wedge\tau_N} X_1^2(s)e^{-\epsilon s}ds \\ &\leq x_1^4 + 6\sigma_1^2\int_0^T E[X_1^2(s)e^{-\epsilon s}]ds \\ &\leq x_1^4 + 6\sigma_1^2\left(x_1^2 + \frac{\sigma_1^2}{\epsilon}\right)t. \end{aligned}$$

Next, we use Hölder's inequality and obtain

$$\begin{aligned} E\left[X_1^2(T\wedge\tau_N)e^{-\alpha(T\wedge\tau_N)}\right] &\leq \left[E\left[X_1^4(T\wedge\tau_N)e^{-\alpha(T\wedge\tau_N)}\right]\right]^{1/2}\left[E\left(e^{-\alpha(T\wedge\tau_N)}\right)\right]^{1/2} \\ &\leq \left[x_1^4 + 6\sigma_1^2\left(x_1^2 + \frac{\sigma_1^2}{\alpha}\right)T\right]^{1/2}\left[E\left(e^{-\alpha(T\wedge\tau_N)}\right)\right]^{1/2}. \end{aligned}$$

By letting N tend to infinity, $\tau_N \to \infty$ and thus

$$\lim_{N\to\infty} E\left[X_1^2(T\wedge\tau_N)e^{-\epsilon(T\wedge\tau_N)}\right] \leq \left[x_1^4 + 6\sigma_1^2\left(x_1^2 + \frac{\sigma_1^2}{\alpha}\right)T\right]^{1/2} e^{-\frac{\alpha}{2}T}.$$

Combining this with (7.93) and (7.94) we obtain

$$\left|\Phi(x_1) - E\int_0^T e^{-\alpha t}X_1^2(t)dt\right| \leq c\left[e^{-\alpha T} + \left[x_1^4 + 6\sigma_1^2\left(x_1^2 + \frac{\sigma_1^2}{\alpha}\right)T\right]^{1/2} e^{-\frac{\alpha}{2}T}\right]$$

By letting T tend to infinity, right hand side tends to zero and the assertion of the lemma follows.

■

Proof of Proposition 4.3: From the representations of $\hat{X}_1^n$, $\hat{X}_2^n$ in (4.53)-(4.54) we get (4.64), using the properties of the Skorohod map defined in Lemma 4.1.

To verify (4.65), first note that from the functional central limit theorem for Poisson processes and Assumption 2.1, we have

$$(\hat{M}_1^n, \hat{M}_2^n, \hat{M}_4^n) \Rightarrow (M_1, M_2, M_4) \text{ as } n \to \infty, \tag{7.95}$$

where M_1, M_2, M_4 are three independent Brownian motion starting from 0, with drift 0 and diffusion parameters $\bar{\lambda}_N, \bar{\mu}, \bar{\lambda}_R$ respectively. Also note that, if we define

$$\zeta^n(t) \doteq \frac{\Phi^n([nt]) - n\beta[nt]}{\sqrt{n}}, \text{ for } t \geq 0, n \geq 1,$$

then it follows that

$$\zeta^n(\cdot) \quad \Rightarrow \quad Z(\cdot), \text{ as } n \to \infty, \tag{7.96}$$

where Z is a driftless Brownian motion, starting from zero with diffusion parameter $\beta(1-\beta)$. From (7.95) it follows that

$$\left(\frac{1}{\sqrt{n}}M_2^n, \frac{1}{\sqrt{n}}M_2^n(\cdot)\right) \quad \Rightarrow \quad (0,0), \text{ as } n \to \infty. \tag{7.97}$$

Also note that from the (4.64) and properties of Skorohod maps in (4.57), (4.56), Assumption 2.4, (4.52) and (7.97) that

$$\frac{1}{\sqrt{n}}\hat{L}_1^n(\cdot) = \psi\left(\frac{x_1^n}{\sqrt{n}} - (\mu^n - \lambda_N^n)e(\cdot) + \left[\frac{1}{\sqrt{n}}M_1^n(\cdot) - \frac{1}{\sqrt{n}}M_2^n(\cdot)\right]\right) \quad \Rightarrow \quad 0, \text{ as } n \to \infty. \tag{7.98}$$

Hence, using Assumption 2.1, we have

$$\theta^n(t) \quad = \quad \frac{1}{n}N_2^n\left(\int_0^t \mu^n \, 1_{\{\hat{X}_1^n(s)>0\}}ds\right) \quad = \quad \frac{1}{\sqrt{n}}\hat{M}_2^n(t) + \mu^n t - \frac{1}{\sqrt{n}}\hat{L}_1^n(t) \quad \Rightarrow \bar{\mu}e(\cdot), \text{ as } n \to \infty \tag{7.99}$$

using (7.97) and (7.98). Therefore, by a random time change theorem (see Sec. 14 of [4]) we obtain

$$\hat{M}_3^n(\cdot) = \zeta^n(\theta^n(\cdot)) \Rightarrow Z(\bar{\mu}\cdot) \doteq M_3(\cdot), \text{ as } n \to \infty, \tag{7.100}$$

where M_3 is a driftless Brownian motion starting from 0 with diffusion parameter $\bar{\mu}\beta(1-\beta)$. It is easy to verify that M_3 is independent of M_1 and M_4 and since $\langle \hat{M}_2^n, \hat{M}_3^n \rangle = -\beta\langle \hat{M}_2^n, \hat{M}_2^n \rangle$, we have from (7.95) that $\langle M_2, M_3 \rangle(t) = -\beta\bar{\mu}t$. Hence, defining $W_1 = M_1 - M_2, W_2 \doteq M_3 - M_4$, we see that $W = (W_1, W_2)$ satisfies the description in (3.19). Also, by the convergence results in (7.95) and

(7.100) as well as the fact that the limits are continuous, we obtain

$$(\hat{W}_1^n, \hat{W}_2^n) \Rightarrow (W_1, W_2), \text{ as } n \to \infty.$$

This completes the proof of (4.65). ■

From (4.65) and using Skorohod embedding theorem, we have that $\hat{W}^n \to W$ almost surely uniformly on compacts, and hence by (4.64) and (4.59), the claim in (4.66) follows. The last statement of the proposition follows trivially from the properties of the Skorohod map defined in Lemma 4.1. ■

Proof of Lemma 4.4 Consider a sequence of random variables $\{Y_n\}$, where for each $n \geq 1$, X_n follows a *Geometric* distribution with parameter $\frac{a_n}{\sqrt{n}}$, and let X be an *Exponential* random variable with parameter a. Since $a_n \to a$ as $n \to \infty$, it is easy to verify that

$$\left(1 - \frac{a_n}{\sqrt{n}}\right)^{\sqrt{n}} \to e^{-a}, \text{ as } n \to \infty.$$

Straightforward calculations using the above yields $\frac{X_n}{\sqrt{n}} \Rightarrow X$. Therefore for all $M > 0$, using the continuity of $c(\cdot)$, we have

$$E\left[h\left(\frac{X_n}{\sqrt{n}}\right) \ 1_{\left\{\frac{X_n}{\sqrt{n}} \leq M\right\}}\right] \to E\left[h(X) \ 1_{\{X \leq M\}}\right] \text{ as } n \to \infty.$$

Hence, for all $M > 0$,

$$a_n \sum_{0 \leq k \leq \sqrt{n}M} h\left(\frac{k}{\sqrt{n}}\right)\left(1 - \frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \to a \int_0^M h(x)e^{-ax}dx, \text{ as } n \to \infty. \tag{7.101}$$

Note that there exists $M_0 > 0$ such that $h(x)e^{-ax}$ decreases as a function of x for all $x \geq M_0$. Fix any $\epsilon > 0$. There exists $M \geq M_0$ such that

$$a \int_M^\infty h(x)e^{-ax}dx < \frac{\epsilon}{3}. \tag{7.102}$$

Since $M \geq M_0$, we have that the integrand below is decreasing and so from (7.102)

$$\sum_{k \geq \sqrt{n}M+1} h\left(\frac{k}{\sqrt{n}}\right)\left(1-\frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \;\leq\; \int_M^\infty h(x)e^{-ax}dx \doteq b_M. \tag{7.103}$$

Fix $n_0(M) \geq 1$ such that $|a_n - a| \leq \epsilon/(3b_M)$, for all $n \geq n_0(M)$. From (7.102) and(7.103) we have for all $n \geq n_0(M)$,

$$\begin{aligned} a_n \sum_{k \geq \sqrt{n}M+1} h\left(\frac{k}{\sqrt{n}}\right)\left(1-\frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \;&\leq\; |\, a_n - a \,| \sum_{k \geq \sqrt{n}M+1} h\left(\frac{k}{\sqrt{n}}\right)\left(1-\frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \\ &\quad + a \sum_{k \geq \sqrt{n}M+1} h\left(\frac{k}{\sqrt{n}}\right)\left(1-\frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \\ &\leq\; |\, a_n - a \,|\, b_M + \; a \int_M^\infty h(x)e^{-ax}dx \;\;\leq\;\; \frac{2\epsilon}{3}. \end{aligned} \tag{7.104}$$

Hence, from (7.102) and (7.104) we have that for all $\epsilon > 0$, there exists large $M > 0$, such that

$$\left| a_n \sum_{k \geq \sqrt{n}M+1} h\left(\frac{k}{\sqrt{n}}\right)\left(1-\frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} - a\int_M^\infty h(x)e^{-ax}dx \right| < \epsilon, \tag{7.105}$$

for $n \geq n_0(M)$. From (7.101) and (7.105), the proof of the lemma is complete. ■