0.0.1 Method of Moments

The Method of Moments is another way of getting estimates for parameters.
The *moments* of a random variable X are defined as the expected values of the powers of X:

Definition 0.0.1 (**Moments**)
The **kth moment** of X is

$$\mu_k := E[X^k] = \int x^k f_X(x)dx.$$

In a lot of distributions these moments of a random variable are fdirectly related to the parameters of the distribution.
Let's for example look at the first moment μ_1 of a random variable X:

$$\mu_1 = \int x f_X(x)dx$$

This is nothing but the expected value of X itself: $\mu_1 = E[X]$.
Now think of the normal distribution N_{μ,σ^2}, we immediately see that $\mu = \mu_1$. For an exponential distribution Exp_λ, we now that the expected value of a random variable with an exponential distribution is equal to $1/\lambda$. Therefore, in this case we see that $\lambda = 1/\mu_1$. For a Poisson distribution with rate λ we see, again, that $\lambda = \mu_1$.
The second moment, μ_2, is related to the variance of a distribution. Remember, that

$$Var[X] = E\left[(X - E[X])^2\right] = E[X^2] - (E[X])^2.$$

Therefore the variance of a distribution is

$$Var[X] = \mu_2 - \mu_1^2.$$

For a normal distribution N_{μ,σ^2} we, again, have the immediate relationship between the variance σ^2 and the moments as:

$$\sigma^2 = \mu_2 - \mu_1^2.$$

Now that we have established the link between moments of a random variable and the parameters of its distribution - how does that help in estimating the parameters?
The idea now, is to estimate the moments of the random variable first - and this is very easy, as we will see - and use the estimates for the moments to get estimates for the parameters by using the link between them.
Estimates for the moments of a random variable X are very easy to get:
The estimator $\hat{\mu_k}$ for the $k$the moment of X is:

$$\hat{\mu_k} := \frac{1}{n}\sum_{i=1}^{n} x_i^k,$$

i.e. we just take the average of the kth power of the observed data values $x_1, x_2, ..., x_n$.

Example 0.0.1 *Parameter Estimation*
The function $F(x) = 1 - x^{-\theta}$ is a distribution function for $x \geq 1$ and $\theta > 0$.
The corresponding density function $f_\theta(x) = \theta x^{-\theta-1}$.
use the Method of Moments to find an estimator for θ:
The first moment μ_1 is the expected value of the above distribution:

$$\begin{aligned}\mu_1 &= \int_1^\infty x \cdot \theta x^{-\theta-1} dx = \int_1^\infty \theta x^{-\theta} dx = \\ &= -\frac{\theta}{-\theta+1} x^{-\theta+1} \,|_1^\infty = -\frac{\theta}{-\theta+1} + 0 = \frac{\theta}{\theta-1}.\end{aligned}$$

We need to solve the equation for θ to get an estimator:

$$\begin{aligned} \mu_1 &= \frac{\theta}{\theta - 1} \\ \iff \mu_1(\theta - 1) &= \theta \\ \iff \theta(\mu_1 - 1) &= \mu_1 \\ \Rightarrow \hat{\theta} = \frac{\mu_1}{\mu_1 - 1}. \end{aligned}$$

Assume that the following ten random numbers come from the above distribution:

$$x_1 = 3.12, x_2 = 1.28, x_3 = 0.53, x_4 = 2.55, x_5 = 1.32, x_6 = 6.64, x_7 = 1.01, x_8 = 0.75, x_9 = 1.07, x_{10} = 0.64$$

Use the Method of Moments to estimate θ:
$\hat{\mu_1}$ is nothing but the average of the above values, $\hat{\mu_1} = 1.891$ and $\hat{\theta} = 2.12$.

Example 0.0.2 *Parameters of the log-normal Distribution*
The log-normal distribution is a continuous distribution with two parameters M and S. Its density function is given as

$$f_{M,S}(x) = \frac{1}{Sx\sqrt{2\pi}} e^{-(\ln x - M)^2/(2S^2)} \text{ for positive } x.$$

1. Show that the first and second moment of the log-normal distribution are

$$\mu_1 = e^{M+S^2/2} \qquad \text{and} \qquad \mu_2 = e^{2(M+S^2)}$$

2. Use the Method of Moments to get estimators for M and S^2.

For the first moment μ_1 we need to compute the integral $\int_0^\infty xf(x)dx$.

$$\mu_1 = \int_0^\infty xf(x)dx = (*)$$

Substituting $\ln x$ by y changes the limits from 0 and ∞ to $-\infty$ and ∞, also $dy = \frac{1}{x}dx$. With that:

$$\begin{aligned} (*) &= \int_{-\infty}^{\infty} e^y \cdot \frac{1}{S\sqrt{2\pi}} e^{-(y-M)^2/(2S^2)} dy = \\ &= \int_{-\infty}^{\infty} \frac{1}{S\sqrt{2\pi}} e^{-(y^2-2My-2S^2y+M^2)/(2S^2)} dy = \\ &= \int_{-\infty}^{\infty} \frac{1}{S\sqrt{2\pi}} e^{-(y^2-M-S^2)/(2S^2)} dy \cdot e^{\frac{(2M+S^2)S^2}{2S^2}} = \\ &= 1 \cdot e^{M+S^2/2} = e^{M+S^2/2}. \end{aligned}$$

Getting the second moment μ_2 is done in the same way, but is a bit lengthier:

$$\int_0^\infty x^2 \cdot f(x)dx = ... = e^{2(M+S^2)}.$$

Now on to the estimators. From the first and second moment, we get two equations for M and S^2:

$$\begin{aligned} (1) &\qquad M + S^2/2 = \ln \mu_1 \\ (2) &\qquad M + S^2 = 0.5 \cdot \ln \mu_2 \end{aligned}$$

Subtracting the two equations gives

$$\begin{aligned} 0.5S^2 &= \ln\mu_1 - 0.5\cdot\ln\mu_2 \\ \Rightarrow \quad & \hat{S}^2 = 2\ln\mu_1 - \ln\mu_2 \end{aligned}$$

Looking at $(1) - 0.5\cdot(2)$ *yields:*

$$\begin{aligned} 0.5M &= \ln\mu_1 - 0.25\cdot\ln\mu_2 \\ \Rightarrow \quad & \hat{M} = 2\ln\mu_1 - 0.5\ln\mu_2 \end{aligned}$$