Practice Problems for the Final Exam - Sample Solution

1. **(Random Movements of particles with absorbing boundary)**

A particle moves randomly among the integer points (positions) $\{1, 2, \ldots, 5\}$. States 1 and 5 are absorbing. At each time point, if the particle is at i, for i = 2, 3, 4, the particle jumps to $i+1$ with probability $i/5$, and otherwise jumps to $i-1$. All such jumps are independent of the past history of the process. The particle starts in state 4.

(a) Write down the transition probability matrix P.
Transition prob matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3/5 & 0 & 2/5 & 0 & 0 \\ 0 & 2/5 & 0 & 3/5 & 0 \\ 0 & 0 & 1/5 & 0 & 4/5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) If the particle is in state 3 now, what is the probability that it will be in state 3 again after 2 jumps?
To answer this we need to compute the (3,3)-th element of the matrix P^2. Which is 0.28

(c) Which states are recurrent and which states are transient?
1, 5 are recurrent (absorbing). All other states are transient

(d) Find the probability that it will be absorbed in state 1. Show all your work.
We set up the system of equations. Let a_i be the probability of absorption in state 1 starting from state i. We just need a_4, since in our problem the chain starts from state 4 - but it is easier to solve for all a_i together by setting up the system of equation.

$$\begin{aligned} a_1 &= 1, \quad a_5 = 0 \\ a_2 &= 3/5 + 2/5\ a_3 \\ a_3 &= 2/5\ a_2 + 3/5\ a_4 \\ a_4 &= 1/5\ a_3 + 0 \end{aligned}$$

Solving it, one gets the following :

$$a_1 = 1, a_2 = 11/15, a_3 = 5/15, a_4 = 1/15, a_5 = 0$$

So, the answer is 1/15

(e) Find the expected number of steps until absorption in $\{1, 5\}$ (i.e, either in 1 or in 5).
We set up the system of equations. Let μ_i be the expected number of steps until absorption (in state 1 or 5) starting from state i. We just need μ_4, since in our problem the chain starts from state 4 - but it is easier to solve for all μ_i together by setting up the system of equation.

$$\begin{aligned} \mu_1 &= 0, \quad \mu_5 = 0 \\ \mu_2 &= 1 + 0 + 2/5\ \mu_3 \\ \mu_3 &= 1 + 2/5\ \mu_2 + 3/5\ \mu_4 \\ \mu_4 &= 1 + 1/5\ \mu_3 + 0 \end{aligned}$$

Solving it, one gets the following :

$$\mu_1 = 0, \mu_2 = 19/9 = 2.1, \mu_3 = 25/9 = 2.8, \mu_4 = 14/9 = 1.6, a_5 = 0$$

So, the answer is 14/9=1.6

2. **(Same particle system with reflecting boundary)**

The above system of particles is modified and now 1 and 5 are **not** absorbing states. If the particle is in state 1, it jumps to state 2 with probability 1. If the particle is in state 5, it jumps to state 4 with probability 1. Otherwise, when the particle is in state i, for i = 2, 3, 4, the particle jumps to $i+1$ with probability $i/5$, and otherwise jumps to $i-1$. All such jumps are independent of the past history of the process.

(a) Write down the transition probability matrix P.
Transition prob matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 3/5 & 0 & 2/5 & 0 & 0 \\ 0 & 2/5 & 0 & 3/5 & 0 \\ 0 & 0 & 1/5 & 0 & 4/5 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) If the particle is in state 3 now, what is the probability that it will be in state 3 again after 2 jumps?
To answer this we need to compute the (3,3)-th element of the matrix P^2. Which is 0.28

(c) Which states are recurrent and which states are transient?
All states are recurrent now

(d) Find the steady state probabilities of the chain.
We set up the system of equations. Let π_j be the steady state probability of being in state j, and $\pi = (\pi_1, \ldots, \pi_5)$. Then we need to solve

$$\pi P = \pi$$

subject to the constraint $\sum_j \pi_j = 1$. Rewriting the above equation we get

$$\begin{aligned} 3/5\ \pi_2 &= \pi_1 \\ \pi_1 + 2/5\ \pi_3 &= \pi_2 \\ 2/5\pi_2 + 1/5\ \pi_4 &= \pi_3 \\ 3/5\ \pi_3 + \pi_5 &= \pi_4 \\ 4/5\ \pi_4 &= \pi_5 \\ \textit{and } \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 &= 1 \end{aligned}$$

Solving it, one gets the following :

$$\pi_1 = 0.0750, \pi_2 = 0.125, \pi_3 = 0.125, \pi_4 = 0.375, \pi_5 = 0.300$$

(e) What is the long term proportion of times the chain spends in state 4?
That will be $\pi_4 = 0.375$

3. **Hourly inspection of computers**

A computer is inspected at the end of every hour. It is found to be either working (up) or fail (down). If the computer is found to be up, the probability of its remaining up for the next hour is 0.9. If it is down, the computer is repaired, which may require more than 1 hour. Whenever the computer is down (regardless of how long it has been down), the probability of its still being down 1 hour later is 0.35.

(a) Construct the (one-step) transition matrix for this Markov chain. *Assuming we order the states as Up and $Down$, we get for the transition matrix:*

$$P = \begin{pmatrix} 0.9 & 0.1 \\ 0.65 & 0.35 \end{pmatrix}$$

Remember, that in a transition matrix the rows have to sum to 1!

(b) What is the steady state probability that the computer is working?

Using the balance equations we get for the steady state probabilities p_U and p_D:

$$\begin{array}{lll} (I) & p_U = & 0.9p_U + 0.65p_D \\ (II) & p_D = & 0.1p_U + 0.35p_D \end{array}$$

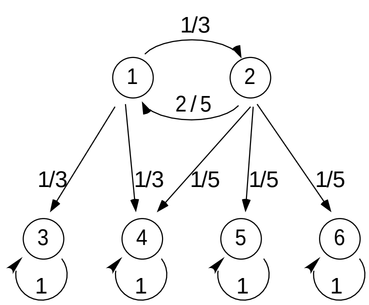
Both equations simplify to $p_U = 6.5p_D$.
With the normalization condition $p_D + p_U = 1$ this yields:

$$\begin{array}{ll} p_U = & 6.5/7.5 = 0.87 \\ p_D = & 1/6.5 = 0.13 \end{array}$$

4. **General Markov Chains** The following matrix is the transition matrix for a Markov chain with states $\{1, 2, 3, 4, 5, 6\}$.

$$P = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 2/5 & 0 & 0 & 1/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Draw a state diagram corresponding to P and find a decomposition into recurrent classes and transient states.



the recurrent classes of this Markov Chain are $\{3\}, \{4\}, \{5\}$, and $\{6\}$; the transient states are 1 and 2.

(b) If the chain starts at state 1 how many steps do you expect will it take until the chain lands in an absorbing state?

Let μ_i be the number of expected steps to reach an absorbing state from i, we have the following set of equations:

$$\begin{aligned}
\mu_1 &= 1 + 1/3\mu_2 + 1/3\mu_3 + 1/3\mu_4 = 1 + 1/3\mu_2 \\
\mu_2 &= 1 + 2/5\mu_1 \\
\mu_3 &= 0 \\
\mu_4 &= 0 \\
\mu_5 &= 0 \\
\mu_6 &= 0
\end{aligned}$$

This gives $\mu_2 = 1 + 2/5(1 + 1/3\mu_2)$. Therefore $\mu_2 = 21/13 = 1.62$ and $\mu_1 = 20/13 = 1.54$ expected steps until absorption.

(c) Again assuming that the chain starts in state 1, find the likelihood of being absorbed in state 6.

Using the absorption probabilities $a_i := P(\text{reach state } 6 \mid \text{ start in } i)$, we get a set of equations:

$$\begin{aligned}
a_1 &= 1/3a_2 + 1/3a_3 + 1/3a_4 = 1/3a_2 \\
a_2 &= 2/5a_1 + 1/5a_6 = 2/5a_1 + 1/5 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 1
\end{aligned}$$

Therefore $a_1 = 1/3(2/5a_1 + 1/5)$, $a_1 = 1/13$. The probability to (eventually) reach state 6 from state 1 is 1/13 = 0.08.

(d) Suppose we start in state 1 with probability 1/3 and in state 2 with probability 2/3 . Find the probability of being in any given state after one step.

To solve this problem, we use the formula for marginal probabilities: $p(1) = p(0)P$, with

$$p_1(0) = 1/3, p_2(0) = 2/3, p_i(0) = 0, \text{ for all } i \neq 1, 2$$

i.e, the initial distribution is

$$p(0) = \left(\frac{1}{3}, \frac{2}{3}, 0, 0, 0, 0\right)$$

With this and the transition matrix P, we get from the formula that

$$p(1) = p(0)P = \frac{1}{3}\left(\frac{4}{5}, \frac{1}{3}, \frac{1}{3}, \frac{11}{15}, \frac{2}{5}, \frac{2}{5}\right)$$

That is

$$P(X_1 = 1) = p_1(1) = \frac{4}{15}, P(X_1 = 2) = p_2(1) = \frac{1}{9} \ldots ..$$