Practice Problems for the Final Exam

1. **(Random Movements of particles with absorbing boundary)**

   A particle moves randomly among the integer points (positions) $\{1, 2, \ldots, 5\}$. States 1 and 5 are absorbing. At each time point, if the particle is at i, for i = 2, 3, 4, the particle jumps to $i+1$ with probability $i/5$, and otherwise jumps to $i-1$. All such jumps are independent of the past history of the process. The particle starts in state 4.

   (a) Write down the transition probability matrix P.

   (b) If the particle is in state 3 now, what is the probability that it will be in state 3 again after 2 jumps?

   (c) Which states are recurrent and which states are transient?

   (d) Find the probability that it will be absorbed in state 1. Show all your work.

   (e) Find the expected number of steps until absorption in $\{1, 5\}$ (i.e, either in 1 or in 5).

2. **(Same particle system with reflecting boundary)**

   The above system of particles is modified and now 1 and 5 are **not** absorbing states. If the particle is in state 1, it jumps to state 2 with probability 1. If the particle is in state 5, it jumps to state 4 with probability 1. Otherwise, when the particle is in state i, for i = 2, 3, 4, the particle jumps to $i+1$ with probability $i/5$, and otherwise jumps to $i-1$. All such jumps are independent of the past history of the process.

   (a) Write down the transition probability matrix P.

   (b) If the particle is in state 3 now, what is the probability that it will be in state 3 again after 2 jumps?

   (c) Which states are recurrent and which states are transient?

   (d) Find the steady state probabilities of the chain.

   (e) What is the long term proportion of times the chain spends in state 4?

3. **Hourly inspection of computers**

A computer is inspected at the end of every hour. It is found to be either working (up) or fail (down). If the computer is found to be up, the probability of its remaining up for the next hour is 0.9. If it is down, the computer is repaired, which may require more than 1 hour. Whenever the computer is down (regardless of how long it has been down), the probability of its still being down 1 hour later is 0.35.

(a) Construct the (one-step) transition matrix for this Markov chain.

(b) What is the steady state probability that the computer is working?

4. **General Markov Chains** The following matrix is the transition matrix for a Markov chain with states $\{1, 2, 3, 4, 5, 6\}$.

$$P = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 2/5 & 0 & 0 & 1/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Draw a state diagram corresponding to P and find a decomposition into recurrent classes and transient states.

(b) If the chain starts at state 1 how many steps do you expect will it take until the chain lands in an absorbing state?

(c) Again assuming that the chain starts in state 1, find the likelihood of being absorbed in state 6.

(d) Suppose we start in state 1 with probability 1/3 and in state 2 with probability 2/3 . Find the probability of being in any given state after one step.