Stat 322 - Homework 8 - Sample Solution

1 Noisy Communication

Suppose that X_n, $n = 1, 2, 3, ...$ is a Bernoulli process with parameter p, i.e. $P(X_n = 1) = p$ for all n.
Suppose that W_n, $n = 1, 2, 3, ...$ is another Bernoulli process with parameter s.
We assume that the two random processes are completely independent of each other (that is, any collection of samples of X_n is independent from any collection of W_n). We form a new random process Y_n, $n = 1, 2, 3, ...$ by defining

$$Y_n = X_n \oplus W_n,$$

where the $\oplus$ operation denotes mod 2 addition. This setup can be thought of as taking an input digital signal X_n and sending it across a binary channel to a receiver. The binary channel can cause an error between the input X_n and output Y_n with probability s. Such a communication channel is called an additive noise channel because the output is the input plus an independent noise process (where plus here means mod 2).

(a) We know that Y_n is a Bernoulli process. Find its parameter, i.e. compute $P(Y_n = 1)$ *The way Y_n is defined we have two possible outcomes for Y_n: 0 and 1.*

$Y_n = 0$ *if* $X_n = W_n$ *and* $Y_n = 1$ *if* $X_n \neq W_n$.

$$\begin{aligned} P(Y_n = 1) &= P(X_n \neq W_n) = P((X_n = 1 \cap W_n = 0) \cup (X_n = 0 \cap W_n = 1)) = \\ &= P((X_n = 1 \cap W_n = 0)) + P((X_n = 0 \cap W_n = 1)) = && \textit{disjoint events} \\ &= P(X_n = 1)P(W_n = 0) + P(X_n = 0)P(W_n = 1) = && X_n, W_n \textit{ independent} \\ &= p(1-s) + (1-p)s = p + s - 2ps \doteq P_y \textit{(say)} \end{aligned}$$

(b) Write down the distribution of the number of signals sent until the k th 1 is sent. Also find the distribution of number of signals received until the kth 1 is received.

Let the random variables be X and Y respectively. Then X follows Negative Binomial with parameters (k, p) and Y follows Negative Binomial with parameters (k, p_y)

(c) If the signal 0 was sent, what is the probability that a 0 is received? a 1 is received?

$$P(Y_n = 0 \mid X_n = 0) = P(X_n \oplus W_n = 0 \mid X_n = 0) = P(0 \oplus W_n = 0) = P(W_n = 0) = 1 - s$$

and, similarly,

$$P(Y_n = 1 \mid X_n = 0) = P(0 \oplus W_n = 1 \mid X_n = 0) = P(W_n = 1) = s$$

(d) Find the conditional pmf $p_{X_n|Y_n}(k \mid j)$.

This problem is more realistically: if we receive a signal Y_n - what is the probability that this was the signal sent - what is the probability that another signal was sent?

We have four probabilities to compute, one for each combination of $X_n = 0, 1$ and $Y_n = 0, 1$:

$$\begin{aligned} p_{X_n|Y_n}(0 \mid 0) &= \frac{P(X_n = 0 \cap Y_n = 0)}{P(Y_n = 0)} = \frac{P(X_n = 0 \cap X_n \oplus W_n = 0)}{P(Y_n = 0)} = \\ &= \frac{P(X_n = 0 \cap W_n = 0)}{1 - P(Y_n = 1)} = \frac{(1-s)(1-p)}{1 - (s + p - 2ps)} = \frac{1 - s - p + ps}{1 + 2ps - s - p}. \\ p_{X_n|Y_n}(1 \mid 0) &= \frac{P(X_n = 1 \cap Y_n = 0)}{P(Y_n = 0)} = \frac{P(X_n = 1 \cap W_n = 1)}{P(Y_n = 0)} = \\ &= \frac{ps}{1 - (s + p - 2ps)} = \frac{ps}{1 + 2ps - s - p}. \end{aligned}$$

$$\begin{aligned}
p_{X_n|Y_n}(0 \mid 1) &= \frac{P(X_n = 0 \cap Y_n = 1)}{P(Y_n = 1)} = \frac{P(X_n = 0 \cap W_n = 1)}{P(Y_n = 1)} = \\
&= \frac{(1-p)s}{s+p-2ps} = \frac{s-ps}{s+p-2ps}. \\
p_{X_n|Y_n}(1 \mid 1) &= \frac{P(X_n = 1 \cap Y_n = 1)}{P(Y_n = 1)} = \frac{P(X_n = 1 \cap W_n = 0)}{P(Y_n = 1)} = \\
&= \frac{p(1-s)}{s+p-2ps} = \frac{p-ps}{s+p-2ps}.
\end{aligned}$$

(e) Find an expression for the probability of error, i.e. $P(Y_n \neq X_n)$.

$$P(Y_n \neq X_n) = P(X_n \oplus W_n \neq X_n) = P(W_n = 1) = s$$

2 Bridges in Madison County

A train bridge is constructed across a wide river. Trains arrive at the bridge according to a Poisson process of rate $\lambda = 3$ per day.

(a) Find probability that the first train arrives after day 1.

X = time for the 1st train to arrive. Then $X \sim Exp(\lambda = 3)$. Hence $P(X > 1) = e^{-3\cdot 1} = e^{-3} = 0.0498$.

(b) Find probability that the 3rd train arrives after 1 days.

Y = time until the 3rd train arrives. Then $Y \sim Erlang(k = 3, \lambda = 3)$. Hence $P(Y > 1) = 1 - P(Y \leq 1) = 1 - Erlang_{3,3}(1) = Po_3(2) = 0.4232$

(c) If one train arrives on day 1, find the probability that there will be no trains on days 2, 3, and 4.

The probability for no train on any given day is $Po_3(0) = e^{-3} \approx 0.05$.

$$P(\text{ no trains on days 2,3,4 } \mid \text{ train on day 1}) = P(\text{ no trains on days 2,3,4 }) = (e^{-3})^3 \approx 0.00012.$$

We could also compute this as

$$P(\text{ no trains on days 2,3,4 } \mid \text{ train on day 1}) = P(\text{ no trains in 3 days }) = Po_{3\cdot 3}(0) = e^{-9} \approx 0.00012.$$

(d) Find the probability that no trains arrive in the first 2 days, but 4 trains arrive on the 4th day.

$$P(\text{ no train on days 1,2 } \cap \text{ 4 trains on day 4 }) \overset{independence}{=} (e^{-3})^2 \cdot e^{-3}\frac{3^4}{4!} = 0.0004.$$

(e) Find the probability that it takes more than 2 days for the 5th train to arrive at the bridge.

We can re-express the above event as: during the first two days we see less than 5 trains.

X := number of trains in 2 days $\sim Po_{3\cdot 2}$, therefore

$$P(X < 5) = Po_6(4) = 0.2851.$$