Stat 322 - Solution to Homework 5

1 Calculators and random numbers

When you generate a random number using your calculator, the random number you get is uniformly distributed over the interval $(0, 1)$. Suppose 50 people in Stat 322 class generate one random number each (independently).

$X =$ number of people that get a random number that is more than 0.98

(a) What is the distribution of X ? Name and write parameters of the distribution or write down the p.m.f
It is Binomial with $n = 50, p = 0.02$

(b) What is the expected number of people to get a random number greater than 0.98?
It is $E(X) = n \cdot p = 50 \times 0.02 = 1$

(c) What is the probability that less than a tenth of the class (i.e 4 or less students) get a random number greater than 0.98?

The answer is same as

$$P(X \leq 4) = B_{50,0.02}(4) \approx Po_1(4) = 0.9963402$$

2 Density Functions

(a) What are the two properties of a probability density function $f(x)$?

A function is a probability density function, if $f(x)$ *is positive (non-negative) for all* x *and the integral over all* x *is 1.*

(b) Which of the following are valid density functions? Explain why or why not, a yes or no is not a sufficient answer.

$$f(x) = \begin{cases} x^2 + 2x, & \text{for } -1 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

$f(x)$ *is no density function, since* $f(-1) = -1$.

$$g(x) = \begin{cases} 2x, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

$g(x)$ *is a density function:* $g(x)$ *is non-negative for all* x, *and*

$$\int_{-\infty}^{\infty} g(x)dx = \int_0^1 2xdx = x^2 \mid_0^1 = 1.$$

(c) Find expected value and variance of the random variable X with distribution function

$$F_X(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-3t} & \text{for } t \geq 0 \end{cases}$$

By looking carefully at the above formula, we can see, that F is an Exponential distribution function with $\lambda = 3$. Expected value and variance of X are therefore 1/3 and 1/9, respectively.

We can get those values by doing the math, too:

$$\begin{aligned} f_X(t) &= F'_X(t) = \begin{cases} 0 & \textit{for } t < 0 \\ 0 + 3e^{-3t} & \textit{for } t \geq 0 \end{cases} \\ E[X] &= \int_{-\infty}^{+\infty} t f_X(t) dt = \int_0^{+\infty} t \cdot 3e^{-3t} dt = (*) \end{aligned}$$

An antiderivative of $3te^{-3t}$ is $-te^{-3t} - \frac{1}{3}e^{-3t}$ (check by differentiating). Then

$$\begin{aligned} E[X] &= -te^{-3t} - \frac{1}{3}e^{-3t} \mid_0^\infty = 0 - (-\frac{1}{3}) = 1/3. \\ Var[X] &= \int_0^\infty (t - 1/3)^2 \cdot 3e^{-3t} dt = ... = 1/9. \end{aligned}$$

3 Continuous Random Variables

For a continuous random variable X the following function is given:

$$f(x) = \begin{cases} k(2 - 2x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find k, so that $f(x)$ is a density function.

For positive k $f(x)$ is non-negative.

Additionally, the integral must be 1.

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 k(2 - 2x)dx = k(2x - x^2) \mid_0^1 = k.$$

Therefore, $k = 1$.

(b) Compute $P(X = 0.5)$.

$$P(X = 0.5) = 0.$$

(c) Compute $P(X \leq 0.5)$.

$$P(X \leq 0.5) = (2x - x^2) \mid_0^{0.5} = 0.75.$$

(d) Find $E[X], Var[X]$.

$$E[X] = \int_{-\infty}^{\infty} x f(x)dx = \int_0^1 x(2 - 2x)dx = x^2 - 2/3x^3 \mid_0^1 = 1/3.$$

$$Var[X] = \int x^2 f(x)dx - (1/3)^2 = 1/18$$

4 Customers in a Bank

In the Ames International Campus Bank (open 24h every day) 5 customers arrive on average during an hour. Answer the following questions. First write down the problem in terms of some random variables and write down what distribution assumptions you make about these random variables.

(a) What is the probability that during an hour no customer arrives?

Let X be the number of customers in one hour; then $X \sim Po_5$.

$$P(X = 0) = e^{-5} \cdot \frac{5^0}{0!} = 0.006738.$$

alternatively, we could have looked at $Y :=$ time until the first customer arrives; then $Y \sim Exp_5$.

$$P(Y > 1) = 1 - P(Y \leq 1) = 1 - (1 - e^{-5 \cdot 1}) = e^{-5} = 0.006738.$$

(b) What is the probability that during an hour, 7 or more customers arrive?

Let X be the number of customers in one hour; then $X \sim Po_5$.

$$P(X \geq 7) = 1 - P(X \leq 6) = 1 - Po_5(6) \stackrel{table}{=} 1 - 0.762183 = 0.237817.$$

(c) What is the probability that there's more than 30 minutes between the 2nd and 3rd customer of the day?

Let Y be the time between the 2nd and the 3rd customer; then $Y \sim Exp_5$ - with $\lambda = 5$ customers per hour.

$$P(Y \geq 0.5) = 1 - P(Y \leq 0.5) = 1 - (1 - e^{-5 \cdot 0.5}) = 0.082085.$$

(d) What is the probability that you have to wait less than an hour for seven customers to arrive?

Let Z be the time until the seventh customer arrives; then $Z \sim Erlang_{7,5}$ - with $\lambda = 5$ customers per hour.

$$P(Z \leq 1) = Erl_{7,5}(1) = 1 - Po_5(6) = 1 - 0.762183 = 0.237817 \quad \textit{compare results with (b) above}$$

(e) How many minutes do you expect to wait until the 12th customer arrives? *Let Z be the time until the twelfths customer arrives; then $Z \sim Erlang_{12,5}$ - with $\lambda = 5$ customers per hour.*

$$E[Z] = 12 \cdot \frac{1}{5} = 2.4 \text{ (hours on average until 12th customer arrives)} = 144 \text{ (min)}.$$

(f) How many minutes do you expect to wait on average between arrivals?

Let Y be the time between arrivals; then $Y \sim Exp_5$ - with $\lambda = 5$ customers per hour.

$$E[Y] = \frac{1}{5} \text{ (of an hour between arrival) } = 12 \text{ min between arrivals.}$$