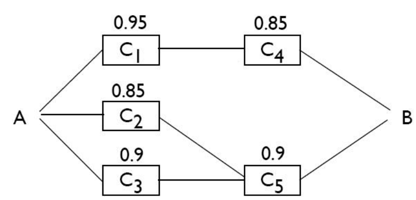
Stat322 - Solution to Homework 2

1 Reliability

Evaluate the reliability of the five component system sketched below, i.e. what is the probability that there is an open path between A and B. The numbers on top of the components give the probability of a properly working component.



The above system works as long as at least one of the paths between A and B work. Note that there are two ways to go from A to C_5 and thats a parallel system. Hence,

$$P(A \to C_5) = 1 - (1 - P(AC_2C_5))(1 - P(AC_3C_5)) = 1 - (0.15)(0.1) = 0.985$$

$$P(AC_5B) = P(A \to C_5)(0.9) = 0.8865$$

$$P(AC_1C_4B) = (0.95)(0.85) = 0.8075$$

Hence,

$$P(A \to B) = 1 - (1 - 0.8075)(1 - 0.8865) = 0.97815125$$

2 Defective Components

A lot of components contains 0.5% defectives. Each component is subjected to a test that identifies a defective component with 99.9% probability. About 2 in 100 good components are also indicated as defective. Given that a randomly chosen component is declared defective by the tester. What is the probability that it actually is defective? - Does the result surprise you? Explain.

Let d be the event that a component is defective,

t_d the event that the test identifies a component as defective (correspondingly t_g = test identifies component as good)

and c_d the event that the component is defective (c_g = good component).

We are given the following probabilities:

$$\begin{aligned} P(c_d) &= 0.005 \\ P(t_d \mid c_d) &= 0.999 \\ P(t_d \mid c_g) &= 0.02 \qquad \textit{rate of false positive test results} \end{aligned}$$

Note: *By comparing the rates of actually defective components and the rate of false positives, we once again see that the test is too strict. The probability that a component that has been identified as being defective will actually be defective will turn out to be fairly small.*

$$P(c_d \mid t_d) \overset{Bayes}{=} \frac{P(t_d \mid c_d)P(c_d)}{P(t_d)} = \frac{0.999 \cdot 0.005}{P(t_d)} =$$

$$\overset{total\ probability}{=} \frac{0.004995}{P(t_d \mid c_d)P(c_d) + P(t_d \mid c_g)P(c_g)} = \frac{0.004995}{0.004995 + 0.02 \cdot 0.995} = \frac{0.004995}{0.024895} \approx 0.20$$

Looking at the previous note the result therefore does not come as a surprise to us.

3 Counting

(a) The Delta Secret Philosophy society has just recruited three football players, two nerds, and four computer geeks. Big Nico picks four of these new members at random for the philosophical discussion team .

  (a) How many different sets of four new members are possible?
  total there are 3+2+4 = 9 new members. Picking 4 from 9 can be done in

  $$\binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{5 \cdot 4 \cdot 3 \cdot 2} = 126$$

  ways.

  (b) How many of those sets of 4 contain both nerds and two computer geeks?
  break the action into two: first pick two nerds, then two geeks. Then we use the multiplication principle:

  $$\underbrace{\binom{2}{2}}_{nerds} \cdot \underbrace{\binom{4}{2}}_{geeks} = 1 \cdot 6 = 6$$

  There are 6 different sets that contain both nerds and two computer geeks.

(b) Egg Hunt: How many different possibilities are there to distribute 20 eggs to three nests?
each egg has a choice of 3 nests:

$$3 \cdot 3 \cdot \ldots \cdot 3 = 3^{20} = 3486784401.$$

different possibilities.

(c) On a party six people shake hands: each one shakes hands with everybody else (only once). How many handshakes does that make in total?
There are two alternative approaches to this problem:

We can work from the basics. The first person shakes hands 5 times, the second shakes hand 4 times, ... = 5+4+3+2+1 = 15 hand shakes total.

The other, more elegant solution is that we think about a handshake as choosing a set of two people. There are $\binom{6}{2}$ *= 15 possibilities to choose 2 out of 6 people.*

(d) A delivery of 50 transistors contains 40 good ones and 10 defectives. In a test five of them are checked. How many possibilities are there to have 3 good ones and 2 defective transistors in the test set?
that's the same as choosing the four new members in question 1b:

$$\underbrace{\binom{40}{3}}_{good} \cdot \underbrace{\binom{10}{2}}_{bad} = 9880 \cdot 45 = 444600.$$

(e) Someone has 15 books - 3 on cooking, 5 on music and 7 novels. How many ways does Someone have to arrange the books on a shelf, if books on the same topic are supposed to be together?

Re-ordering k objects can be done in $k!$ different ways (that's $P(k,k)$). Therefore we have

$$\underbrace{3!}_{cooking} \cdot \underbrace{5!}_{music} \cdot \underbrace{7!}_{novels} \cdot \underbrace{3!}_{topics} = 21772800.$$

different arrangements of books

(f) How many ways are there to arrange the letters M I S S I S S I P P I to different "words"?

this problem can be solved in different ways, maybe easiest is the (once again) by using the multiplication principle. We break down the problem into four parts: choosing a position for M, then four positions for the Is, then 4 for the Ss and lastly two positions remain for the Ps.

letter	*#possibilities*
1 M	11
4 Is	$\binom{10}{4}$
4 Ss	$\binom{6}{4}$
2 Ps	1

This gives

$$11 \cdot \binom{10}{4} \cdot \binom{6}{4} \cdot 1 = 11 \cdot \frac{10!}{4! \cdot 6!} \cdot \frac{6!}{4! \cdot 2!} \cdot 1 = \cdot \frac{11!}{4! \cdot 4! \cdot 2} = 34650$$

different arrangements.

4 Break in

A hacker wants to break into a computer, which is password-protected. Assume that there are 300 equally likely passwords and that the hacker chooses passwords independently at random and tries them.
Determine the probability that the hacker is successful in exactly the 5th attempt if

a) unsuccessful passwords are not eliminated from further selections,

b) unsuccessful passwords are eliminated.

Here is the answer:

(a) $(299/300)^4(1/300) = 0.003289$

(b)$(299/300)(298/299)(297/298)(296/297)(1/296) = 0.003333$

5 Communication Channel

A communication channel receives independent pulses at the rate of 12 pulses per microsecond (12 μs^{-1}).
The probability of a transmission error is 0.001 for each pulse. Compute the probabilities of

a) no errors per microsecond,

b) (exactly) one error per microsecond,

c) at least one error per microsecond,

d) exactly two errors per microsecond.

From the perspective of each pulse this is a Bernoulli experiment with outcomes 'no error' and 'error'. The probability that no error occurred is 1-0.001 = 0.999.
This experiment is repeated 12 times in a microsecond - which gives us 12 independent repetitions, which makes it a 12-Bernoulli-sequence.

a) $P(\text{ no error in a micorsecond }) = (1 - 0.001)^{12} \approx 0.988$

b) $P(\text{ one error in a microsecond }) = \binom{12}{1}(1 - 0.001)^{11} \cdot 0.001 \approx 0.0119.$

c) $P(\textbf{ at least } \text{one error in a microsecond }) = 1 - P(\text{ no error in a micorsecond }) \approx 0.012$

d) $P(\text{ two errors in a microsecond }) = \binom{12}{2}(1 - 0.001)^{10} \cdot 0.001^2 \approx 6.53 \cdot 10^{-5}$