Stat322 - Solution to Homework 1

1 Broken Components

Four components are inspected and three events are defined as follows: A : "all four components are found to be defective"
B : "exactly two components are found defective"
C : "at most three components are found defective"

Interpret the following events. Start by defining a suitable sample space Ω::

a) $B \cup C$.

b) $B \cap C$.

c) $A \cup C$.

d) $A \cap C$.

A suitable choice for the sample space, is to count the number of defective components. Then

$$\Omega = \{0, 1, 2, 3, 4\}$$

and the events A, B, C *can be rewritten as* $A = \{4\}, B = \{2\}$ *and* $C = \{0, 1, 2, 3\}$
With that

a) $B \cup C = \{2\} \cup \{0, 1, 2, 3\} = C =$ *"at most three components are found defective".*

b) $B \cap C = \{2\} = B =$ *"exactly two components are found defective".*

c) $A \cup C = \{0, 1, 2, 3, 4\} =$ *"any number of components is found to be defective"* $= \Omega$.

d) $A \cap C = \emptyset$.

2 Playing Sports

Out of a group of 40 students all play at least one of badminton, volleyball or table tennis.

8 students play all three games, 10 students play badminton and table tennis

20 students play table tennis and volleyball, 12 students play badminton and volleyball

30 students play table tennis, 25 students play volleyball.

Draw a Venn Diagram of the situation.

(a) How many of the students play *only* badminton?

(b) How many of the students play badminton?

Assume, one student is picked at random out of this group. What is the probability that he/she

(c) plays badminton?

(d) plays at least two sports (out of badminton, volleyball and table tennis)?

Assume the student you've picked is a volleyball player.

(e) What is now the probability that he/she is a badminton player?

(f) Why is the probability for picking a badminton player in (e) different from the probability in (b)?

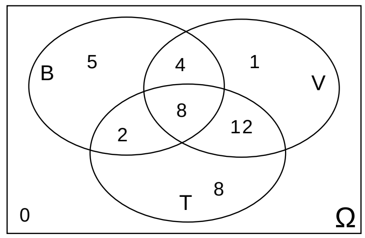
With events $V, B,$ and T defined as:

V *:= a student plays volleyball,*

B *:= a student plays badminton,*

T *:= a student plays table tennis,*

a Venn Diagram of the above situation looks like:



With that, we can answer questions (a) and (b):

(a) *the number of students playing badminton only is 5.*

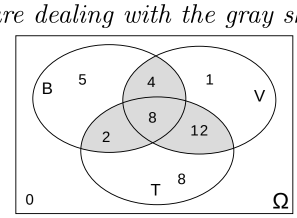
(b) *19 students play badminton.*

Since a student is picked at random, each student has the same probability to be picked. We can assume a sample space $\Omega = \{s_1, s_2, s_3, ..., s_{40}\}$ and the events $V, B,$ and T are subsets of students playing this particular sport.

(c) *the probability that the student plays badminton therefore is*

$$P(B) = \frac{|B|}{|\Omega|} = \frac{19}{40} = 0.475.$$

(d) *the probability that he plays at least two sports is the probability to play volleyball and badminton, or badminton and table tennis, or table tennis and volleyball or all three sports. by looking at the venn diagram, it's easy to see, that we are dealing with the gray shaded area:*



Yielding a probability of $P($ *at least two sports* $) = \frac{4+8+2+12}{40} = 0.65.$

Using a set notation, it's slightly more difficult to come up with the same result:

$$\begin{aligned} P(\text{ at least two sports }) &= P((V \cap \bar{B}\bar{T}) \cup (\bar{\mathcal{V}} \cap \mathcal{B} \cap \bar{\mathcal{T}}) \cup (\bar{\mathcal{V}} \cap \bar{\mathcal{B}} \cup \mathcal{T}) \\ &= P(V \cap B) + P(V \cap T) + P(B \cup T) - 2P(B \cap T \cap V) = \\ &= 0.3 + 0.5 + 0.25 - 2 \cdot 0.2 = 0.65. \end{aligned}$$

Now we found out that student picked is a volleyball player. That changes the probability, that he is a badminton player, too:

(e) *we* know *the student is a badminton player, so we are reducing our sample space to V only (he's one out of 25 volleyball players), 12 out of the 25 volleyball players also play badminton. Therefore the probability, that he plays badminton is 12/25 = 0.48.*

(f) *the different probabilities for playing badminton are due to our gain in knowledge. For the second part, we already knew more, we therefore were able to reduce the sample space. From Last week's lecture we know, that question (e) asks for a conditional probability: the conditional probability of playing badminton given that a student plays volleyball.*

$$P(B \mid V) = \frac{P(B \cap V)}{P(V)} = \frac{0.3}{0.625} = 0.48.$$

3 Sample Spaces

Suggest suitable sample spaces, and identify the subset corresponding to the event A, for the following situations:

(a) A coin, which can show heads (H) or tails (T), is tossed three times; A is the event that the coin shows heads twice.

(b) A game of football is played; A is the event that the match is drawn.

(c) A couple have two children; A is the event that both are girls.

(d) A shot hits a circular target of radius 10 centimetres; A is the event that the shot hits within 3 centimetres of the centre of the target.

(e) A money value x is chosen at random from the financial pages of a newspaper; A is the event that the first digit of x is 1.

In each case discuss what the probability $P(A)$ might be, commenting on how precise you can be, and on any assumptions you need to make.

(a) *tossing a coin three times: the sample space with the highest resolution is*

$$\Omega_a = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

This makes A = (at least) two heads:

$$A = \{HHH, HHT, HTH, THH\}$$

The sample space has the advantage that, using a fair coin, each elementary event is equally likely. Therefore we can compute the probability for A by counting elementary events:

$$P(A) = \frac{4}{8} = 0.5.$$

The main assumption here is, that the coin is fair, i.e. $P(H) = P(T) = 0.5$.

(b) *outcome of a football game:*

$$\Omega_b = \{\textit{home wins, home loses, game drawn}\}$$

The event A = game drawn can be written as:

$$A = \{\textit{ game drawn }\}$$

But we can not say anything about the probability for A, except that it is between 0 and 1 and depends on the teams playing.

(c) *A couple has two children: assuming, that the order reflects the age of the children, we can write a sample space as:*

$$\Omega_c = \{BB, BG, GB, BB\}$$

The event A = two girls is

$$A = \{GG\}$$

To compute the probability of A, we will assume, that the probability to have a boy or a girl are equal, i.e. $P(G) = P(B)$*. That makes all four elementary events of* Ω_c *equally likely, therefore*

$$P(A) = \frac{1}{4} = 0.25.$$

Restrictions: we made the assumption, that the probabilities to have a boy or a girl are equal. In reality, the probabilities might be slightly different, influenced e.g. by climate or culture.

(d) *hit of a circular target: let* Ω_d *describe the distance of the hit from the centre:*

$$\Omega_d = [0, 10]cm$$

Event A = shot hits within 3 cm of center= [0,3] cm.

Under the assumption, that the shot hits randomly, we have to look at the area *event A covers compared to the total area of* Ω_d *(Remember, the area of a circle is computed as* $r^2\pi$*, where r is the radius of the circle):*

$$P(A) = \frac{3^2\pi}{10^2\pi} = 0.09.$$

The assumption to hit the the area randomly might have to be adjusted, if we know more about the shooting skills of the person doing the shooting.

(e) *money value chosen from newspaper:* Ω_e *is the first digit (all digits possible) of a money value.*

$$\Omega_e = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

The event A = first digit is 1 = $\{1\}$*. One assumption in order to get a probability of A is, again, equality. If all ten digits are equally likely,* $P(A) = 0.1$*.*

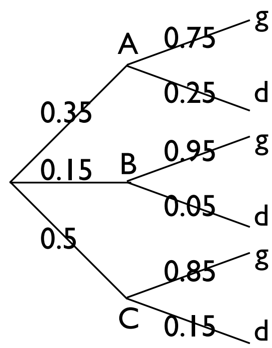
Please note that the above sample spaces are suggestions only. Other solutions might be equally valid.

4 Defective Products

Plants A, B, C produce 35% , 15% and 50% of the total output. Nondefective output is 75% , 95% and 85% respectively.

(a) What is the probability for a customer to receive a defective product?

We can draw a tree of the situation:



The tree corresponds to the probabilities

$$P(A) = .35, P(B) = .15, P(C) = .50 \text{ and } P(g|A) = .75, P(g|B) = .95, P(g|C) = .85 \text{ and } P(d|A) = .25, P(d|B) = .05, P(d|C)$$

Th probability to receive a defective product now corresponds to the three leaves in the tree $P(A \cap d), P(B \cap d)$ *and* $P(C \cap d)$*. We get the probability for a leaf by multiplying every probability on the edges from the root to that leaf (first rule of trees). This gives:*

$$\begin{aligned} P(A \cap d) &= 0.35 \cdot 0.25 = 0.0875 \\ P(B \cap d) &= 0.15 \cdot 0.05 = 0.0075 \\ P(C \cap d) &= 0.50 \cdot 0.15 = 0.075 \end{aligned}$$

In total, we have a probability of 0.0875+0.0075 + 0.075 = 0.17 for a defective product (second rule of trees).

(b) A customer receives a defective product. What is the probability that it came from plant C? *What is* $P(C \mid d)$*?*

By definition we know that $P(C \mid d)$ *is equal to the fraction of* $P(C \cap d)$ *and* $P(d)$*. Therefore:*

$$P(C|d) = \frac{P(C \cap d)}{P(d)} = \frac{0.075}{0.17} \approx 0.44$$