AN ABELIAN LIMIT APPROACH TO A SINGULAR ERGODIC CONTROL PROBLEM

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Abstract
We consider an ergodic stochastic control problem for a class of one-dimensional Itô processes where the available control is an added bounded variation process. The corresponding infinite horizon discounted control problem is solved in [28]. Here, we show that, as the discount factor approaches zero, the optimal strategies derived in [28] “converge” to an optimal strategy for the ergodic control problem. Under different assumptions, two types of optimal strategies were derived. Also, the Abelian limit relationships between the ergodic control problem, the infinite horizon discounted control problem and the finite time horizon control problem are established here. A solution to a constrained optimization problem is obtained as an application.

Keywords: ergodic control, local-time processes, diffusions with reflecting boundaries.

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1 Introduction

Consider a weak solution to the one-dimensional stochastic differential equation

\[ X_x(t) = x + \int_0^t \mu(X_x(s-))ds + \int_0^t \sigma(X_x(s-))dW(s) + A(t) \quad (1.1) \]

where \( x \) is a real number, \( \{W(t) : t \geq 0\} \) is a standard Brownian motion adapted to a right continuous filtration \( \{\mathfrak{F}_t : t \geq 0\} \) on a probability space \((\Omega, \mathfrak{F}, P)\). The \( \sigma \)-algebra \( \mathfrak{F}_0 \) contains all the null sets in \( \mathfrak{F} \) and the Brownian increments \( W(t+s) - W(t) \) are independent of \( \mathfrak{F}_t \) for all \( t \geq 0 \) and \( s \geq 0 \). The control process \( A(\cdot) \) is \( \{\mathfrak{F}_t\} \)-adapted, right continuous with left limits, and of bounded variation on finite time intervals. Also, \( A(0)=0 \).

We further assume that there is a \( \delta_0 > 0 \) so that for each \( X_x(\cdot) \) and \( 0 < \alpha < \delta_0 \), there exists a sequence of stopping times \( (\tau_n) \) satisfying \( \lim_{n \to \infty} \tau_n = \infty \) and

\begin{align*}
(i) \quad & E_x \int_0^{T \wedge \tau_n} [|\mu(X_x(s-))| + \sigma^2(X_x(s-))]ds < \infty, \quad \text{for each } T > 0, \text{ and} \\
(ii) \quad & \lim_{n \to \infty} E_x[|X_x(\tau_n)|e^{-\alpha \tau_n}I_{[\tau_n < \infty]}] = 0. \quad (1.2)
\end{align*}

The first condition helps us to make sense of (1.1) and the second condition will be used in verifying the Abelian limits described below. Throughout this article, we closely rely on several results obtained in [28] and the above two conditions imply the assumption (1.2) in [28].

The quintuple \((\Omega, \mathfrak{F}, P), (\mathfrak{F}_t), W, A, X_x)\) is called an admissible control system if the corresponding state process \( X_x \) satisfies (1.1) and (1.2). Let \( \mathcal{U} \) denote the collection of all such admissible systems and \( C : \mathbb{R} \to \mathbb{R} \) be a running cost function. We shall study the ergodic stochastic control problem with optimal value \( \lambda_0 \) defined by

\[ \lambda_0 \triangleq \inf_{\mathcal{U}} \limsup_{T \to \infty} \frac{1}{T} E_x \int_0^T [C(X_x(s))ds + d|A|(s)]. \quad (1.3) \]

Notice that \( \lambda_0 \) is a constant which is independent of the initial value \( x \), since an initial jump does not alter the above lim sup value for a given state process \( X_x \).

Our goal here is to characterize an optimal control with a Markovian state process \( X \) that achieves the value \( \lambda_0 \) and to relate it with the value functions.
of the family of discounted control problems defined by

$$V_\alpha(x) \equiv \inf_{U} E_x \int_0^\infty e^{-\alpha s} [C(X_x(s))ds + d|A(s)|]$$  \hspace{1cm} (1.4)

as well as to the value functions of the family of finite horizon control problems

$$V_0(x, T) \equiv \inf_{U} E_x \int_0^T [C(X_x(s))ds + d|A(s)|].$$  \hspace{1cm} (1.5)

It turns out that under the assumptions (1.6)-(1.8) below, $V_0(x, T)$ remains the same even when the infimum is taken over all processes $X_x(\cdot)$ which satisfy (1.1) together with the condition $E|X_x(t)| < \infty$ for each $t$ in $[0, T]$. This is because we can extend such a process $X_x(\cdot)$ to $[0, \infty)$ as an admissible process by taking $A(t) \equiv 0$ for all $t > T$, and then using the results outlined in section 3 below.

Throughout this article, we make the basic assumptions (1.6), (1.7) and (1.8) below. Here $\mu', \sigma'$ and $C'$ denote the derivatives of $\mu$, $\sigma$ and $C$ respectively.

(i) The functions $\mu$ and $\sigma$ are continuously differentiable on $R$, $\mu'(y) \leq 0$ for all $y$, $\inf_{R} \sigma(y) > 0$ and $x\mu(x) < 0$ for all $x \neq 0$. \hspace{1cm} (1.6)

(ii) $\int_{-\infty}^{0} \frac{\mu(x)-x}{\sigma^2(x)} dx = \int_{0}^{\infty} \frac{x-\mu(x)}{\sigma^2(x)} dx = \infty$. \hspace{1cm} (1.7)

(iii) The cost function $C$ is continuously differentiable on $R$, decreasing on $(-\infty, 0)$, increasing on $(0, \infty)$, $C(0)=0$ and satisfies one of the following conditions:

Either (a) $\liminf_{|x| \to \infty} \frac{C(x)}{|x|} > 0$ or (b) $\limsup_{|x| \to \infty} \frac{C(x)}{|x|} < \infty$. \hspace{1cm} (1.8)

The condition $C(0) = 0$ is made for convenience. If $C(0)$ is any other value, then it only shifts the value functions of (1.3)-(1.5) by the appropriate constant. The diffusion coefficient $\sigma(\cdot)$ is allowed to be an unbounded function subject to the above conditions (1.6) and (1.7).

Under assumption (1.6), the ordinary differential equation $\dot{x} = \mu(x)$ has a unique global, asymptotically stable equilibrium point at the origin. The cost function $C(\cdot)$ also has its unique minimum at the origin and it increases as $x$ moves away from the origin. Therefore, our study concerns the long term stability of a randomly perturbed stable dynamical system with a minimal control effort.

The qualitative nature of the optimal policies depends on the growth rates of
Therefore, we introduce a function $H$ defined by

$$H(x) = \mu'(x) + |C'(x)| \quad \text{for all } x \in \mathbb{R}. \quad (1.9)$$

The basic relationships between $\lambda_0, V_\alpha(x)$, and $V_0(x,T)$ are known as the “Abelian limit relations” (see [13]). These relations (which hold uniformly over the compact sets) are described by

$$a) \lim_{\alpha \to 0} \sup_{|x| \leq K} |\alpha V_\alpha(x) - \lambda_0| = 0 \quad \text{and} \quad b) \lim_{T \to \infty} \sup_{|x| \leq K} \left| \frac{V_0(x,T)}{T} - \lambda_0 \right| = 0 \quad (1.10)$$

for each $K > 0$.

We will establish these limits in this article. In an interesting article [13], Karatzas derived optimal strategies for (1.3), (1.4), (1.5) and established the Abelian relations (1.10), when the drift coefficient $\mu$ is identically zero and the diffusion coefficient $\sigma$ is a constant. In the symmetric case, where $\mu$ is an odd function and $\sigma$ and $C$ are even functions, an optimal strategy for (1.3) is obtained in [27]. In both of these articles, first a smooth solution to the corresponding Hamilton-Jacobi-Bellman (HJB) equation is obtained and then a verification lemma is used to prove the optimality of the chosen candidate for optimal control.

Here we develop a different approach. Using the properties of the value function $V_\alpha(x)$ of (1.4) derived in [28], we first establish that $\lim_{\alpha \to 0} \alpha V_\alpha(x)$ exists and is equal to a constant $\Lambda_0$ and we also show that $\Lambda_0 \leq \lambda_0$, where $\lambda_0$ is the value of (1.3). Finally, to describe the derivation of an optimal strategy, let the initial point be at the origin. In this case, the optimal state process for $V_\alpha(0)$ derived in [28] induces a probability measure $\nu_\alpha$ on $C[0,\infty)$. Under our assumptions, $\nu_\alpha$ converges weakly (through a subsequence) to a probability measure $\nu_0$ on $C[0,\infty)$ as $\alpha$ tends to zero. Thereafter, we derive an admissible strategy $((\Omega, \mathcal{F}, P), (\mathcal{G}_t), W, A^*, X^*_0)$ in $\mathcal{U}$ so that the corresponding state process $X^*_0$ induces the measure $\nu_0$ on $C[0,\infty)$. It turned out that the corresponding value for the ergodic cost criteria is indeed $\Lambda_0$. Hence, $\Lambda_0$ is equal to the optimal value $\lambda_0$ and the above strategy $((\Omega, \mathcal{G}, P), (\mathcal{G}_t), W, A^*, X^*_0)$ is optimal for (1.3). A complete solution to a constrained optimization problem is derived in section 6 by applying the results in section 4.

In [3], Arisawa and Lions considered an ergodic stochastic control problem in a compact state space. Hence, their cost function is bounded. In their model, there is no added bounded variation process, but the drift and diffusion coefficients are controlled. Their aim was to analyze the solution to HJB equation...
and to establish the uniform convergence of the Abelian limits described in (1.10). But they did not derive any optimal strategies. This work complements their results and shows that uniform convergence on the compact sets for the Abelian limits in (1.10) is the best possible when the state space is non-compact.

In [16], the authors used a controlled martingale formulation for the cost structures in (1.3), (1.4) and (1.5), when there is no added bounded variation process. They related these problems with linear programming problems over a space of measures and prove the existence of optimal Markovian controls. The problem (1.4) is considered in [17], when the drift term is linear and the cost function is convex. There, the optimal control is a local-time process which is similar to our results in section 4. In a series of articles [1], [2], Alvarez considered singular control problems for diffusions with an absorbing barrier at the origin and with an added increasing control process. He used the connection between stochastic control and optimal stopping to derive optimal strategies. In [8], the existence theorems for optimal policies for the ergodic control problem of multidimensional diffusions were developed. A higher dimensional singular control problem for the standard Brownian motion was treated in [24]. For a discussion on singular control problems and related references, we refer to [11].

This article is organized as follows: Section 2 gathers the preliminary results regarding the value functions defined in (1.4) and (1.5). Our main theorem of section 3 is Theorem 3.1. It shows that under certain assumptions, zero control policy is optimal for the ergodic control problem (1.3). It also establishes the Abelian relation (1.10a). In section 4, under a different set of assumptions, we derive an optimal control policy which can be described in terms of local-time processes. The corresponding optimal state process is a reflecting diffusion on a finite interval. We also establish the limit (1.10a) for this case. Section 5 is devoted to the proof of the Abelian limit (1.10b). We apply the results obtained in section 4 to find an optimal policy for a constrained minimization problem in section 6. The main results in this article are Theorems 3.1, 4.1, 5.1 and 6.3. Next, we describe a motivating example from finance.

Example 1. (Foreign exchange rates)
Consider the currency exchange rate that governs the transactions between two countries (which we label as “domestic” and “foreign”). We assume that the economies of both countries are stable. Hence, in the absence of interventions, the currency exchange rate resembles a dynamical system which fluctuates around a stable equilibrium point. In the presence of uncertainty, it is common practice to model currency exchange rates using stochastic differential
equations. (See chapter 7 of [21] and also [10], [12], [15], [19], [20], [25], [26])

Here we consider the problem of a central bank which would like to keep the exchange rate as close as possible to a target value through minimal intervention.

In a pioneering work [15], Krugman introduced a model where the exchange rate takes values in an exogenously given target interval which is commonly called the “target zone” or the “target band”. In recent years, exchange rate target zones have been an area of intense research activity in finance ([10], [12], [19], [20], [25], [26]). When the exchange rate is high, the central bank may intervene by reducing the money supply by selling the foreign currency reserves or by adjusting the domestic interest rates. The central bank may also intervene appropriately when the exchange rate is low. To keep the exchange rate within the target band, the central bank may intervene while the exchange rate is still within the target band and there is empirical evidence to support this claim [7]. Such an optimal intervention policy with jumps within the target band is derived for a target zone exchange rate model using impulse controls in [12].

For a detailed discussion of Krugman’s model, its underlying assumptions, its implications, drawbacks, and for modifications to accompany empirical data, we refer to [26]. One important prediction of Krugman’s original model is that the long term distribution of the logarithm of the exchange rate within the target band must be U-shaped. This implies that in the long run, the logarithm of the exchange rate must spend most of its time near the end points of the target band. But, empirical data rejected this fact, and as described in ([26]), there is a “hump” shaped distribution within the band. In a detailed discussion about an extension of Krugman’s model to agree with data, Svensson [26] pointed out that the logarithm of the exchange rate within the target band displays a “mean reversion behavior” and this is an important property of target zone exchange rates. This property also supports the “hump” shaped long run distribution of the logarithm of the exchange rate.

In [20], a discrete impulse control intervention coupled with a continuous domestic interest rate control policy is used to derive an optimal target band. In [10], a similar problem for a geometric Brownian motion with a cost function $C(x) = x^2$ is considered. The authors also allow a fixed cost and a cost proportional to the size of the intervention. They derive an optimal intervention policy and the explicit form of the value function.
In our model, we consider a target value or a benchmark for the exchange rate which we simply assume to be at 1. The controlled state process \( X_x(\cdot) \) represents the logarithm of the exchange rate. The central bank would like to keep the \( X_x(\cdot) \) process near the origin. Here, there is no a priori assigned target band. We assume that \( X_x(\cdot) \) satisfies (1.1) and the drift and diffusion coefficients \( \mu \) and \( \sigma \) satisfy (1.6) and (1.7). Hence, \( X_x(\cdot) \) clearly shows the mean reversion behavior around the origin as observed in [26]. The bounded variation control process \( A(\cdot) \) represents the changes in the exchange rate due to central bank interventions. The corresponding total variation process \( |A(\cdot)| \) represents the cumulative cost incurred by the central bank interventions. There is also a running cost associated with the deviation of the exchange rate from its benchmark and this running cost function satisfies the assumption (1.8).

With our model, the central bank would like to know the answers to the following two important questions:

1. What optimal intervention policy will minimize the long term average cost criteria? Furthermore, up to what extent will such an optimal intervention policy verify the validity of a target band for the exchange rate?

2. Suppose that the central bank insists on the intervention policy not to exceed the long term average intervention cost above a given target value \( m > 0 \) (i.e. \( \limsup_{T \to \infty} \frac{E[A(T)]}{T} \leq m \)). What would be an optimal intervention policy which minimizes the long term average running cost \( \limsup_{T \to \infty} \frac{1}{T} E \int_0^T C(X_x(s)) \, ds \) ? Under what conditions on \( \mu \), \( \sigma \) and \( C(\cdot) \) does this constrained problem lead to a target zone model?

Sections 3 and 4 provide answers to question 1. Question 2 will be analyzed in section 6. In section 3, we derive a set of sufficient conditions for the optimality of zero intervention policy. Hence, there is no optimal target band under these conditions. With a different set of assumptions on \( \mu \), \( \sigma \) and \( C \), we prove the existence of an optimal target band in section 4. We also derive an optimal intervention policy. In section 6, under the given constraint on the long term average intervention rate, we provide an optimal target band and also an optimal intervention policy.
2 Preliminaries

Here we develop some results related to the value functions of the control problems in (1.3), (1.4), and (1.5). They will be important in establishing Abelian limits in (1.10). Throughout this section, we assume the conditions (1.6), (1.7) and (1.8).

Proposition 2.1 Let $\lambda_0, V_\alpha(x)$ and $V_0(x, T)$ be as in (1.3), (1.4) and (1.5). Then

(i) for each $T > 0$, and $0 < \alpha < \delta_0$, the quantities $\lambda_0$, $V_\alpha(x)$ and $V_0(x, T)$ are finite. Also, for each $K > 0$, there is a constant $M_K$ so that

$$\sup_{0<\alpha<\delta_0} \sup_{|x|\leq K} \alpha V_\alpha(x) \leq M_K.$$  

(ii) for each $T > 0$, $0 < \alpha < \delta_0$ and $x, y$ in $\mathbb{R}$, $|V_\alpha(x, T) - V_\alpha(y, T)| \leq |x - y|$ and $|V_\alpha(x) - V_\alpha(y)| \leq |x - y|$.

(iii) $\limsup_{\alpha \to 0} \alpha V_\alpha(x) \leq \lambda_0$ and $\limsup_{T \to \infty} \frac{1}{T} V_0(x, T) \leq \lambda_0$, where $\lambda_0$ is the value of the ergodic control problem (1.3).

Proof. Given $K > 0$ and $|x| < K$, we pick an interval $[a, b]$ so that $a < -K < K < b$. Consider the reflected diffusion process on $[a, b]$, which is given by

$$X_x(t) = x + \int_0^t \mu(X_x(s))ds + \int_0^t \sigma(X_x(s))dW(s) + L_a(t) - L_b(t). \quad (2.1)$$

Here $L_a$ and $L_b$ are local time processes of $X_x$ at $a$ and $b$ respectively. In comparison with (1.1), $A(t) = L_a(t) - L_b(t)$ and $|A(t)| = L_a(t) + L_b(t)$ for all $t \geq 0$. Consider the solution to the differential equation

$$\frac{\sigma^2(x)}{2} Q''(x) + \mu(x) Q'(x) = \gamma \quad \text{for all} \quad x \in (a, b) \quad \text{and} \quad Q'(a) = -1, \quad Q'(b) = 1 \quad (2.2)$$

where $\gamma > 0$ is a constant which will be chosen appropriately. Notice that (2.2) is a first order equation in $Q'(\cdot)$ and it can be solved using the boundary condition $Q'(a) = -1$. Then, for each $x$ in $[a, b]$, we obtain

$$Q'(x)e^{2\int_a^x \rho(u)du} + e^{-2\int_a^x \rho(u)du} = \gamma \int_a^x \frac{2}{\sigma^2(y)} e^{2\int_a^y \rho(u)du} dy.$$
In the above equation, \( \rho(x) = \frac{\mu(x)}{\sigma^2(x)} \) for all \( x \) in \([a, b]\). Now we choose the positive constant \( \gamma \) so that it satisfies
\[
\gamma \int_a^b \frac{2}{\sigma^2(y)} e^{2 \int_a^b \rho(u)du} dy = e^{2 \int_a^b \rho(u)du} + e^{-2 \int_a^b \rho(u)du},
\] (2.3)
then it enforces \( Q'() \) to satisfy the other boundary condition \( Q'(b) = 1 \) in (2.2). The solution to (2.2) is unique up to a constant and we consider the solution
\[
Q(x) = \int_a^x u(y)dy
\]
where
\[
u(x) = e^{-2 \int_a^x \rho(u)du} \left[ \gamma \int_a^x \frac{2}{\sigma^2(y)} e^{2 \int_a^b \rho(u)du} dy - e^{-2 \int_a^b \rho(u)du} \right].
\]
Here the constant \( \gamma \) satisfies the equation (2.3). Next, we apply Itô’s lemma to \( Q(X_\alpha(t)) \) and obtain
\[
E|A|(T) = E[L_a(T) + L_b(T)] = \gamma T + Q(x) - E[Q(X_\alpha(T))].
\] (2.4)
At this point, using (2.3) together with (2.4), we can derive the following limit (independent of the initial point \( x \))
\[
limit_{T \to \infty} \frac{E|A|(T)}{T} = \gamma = \frac{e^{2 \int_a^b \rho(u)du} + e^{-2 \int_a^b \rho(u)du}}{\int_a^b \frac{2}{\sigma^2(y)} e^{2 \int_a^b \rho(u)du} dy}
\] (2.5)
and it will be used in section 6.
Let \( M_1 > 0 \) be a constant so that
\[
\sup_{[a,b]} ||C(x)|| + ||Q(x)|| < M_1.
\] (2.6)
This combined with (2.4) yields \( E|A|(T) < \gamma T + 2M_1 \). Therefore, we can conclude \( V_0(x, T) \leq (M_1 + \gamma)T + 2M_1 \) and
\[
\lambda_0 \leq \limsup_{T \to \infty} \frac{1}{T} E \int_0^T [C(X_\alpha(s))ds + d|A|] \leq (M_1 + \gamma). \]
Consequently, \( V_0(x, T) \) and \( \lambda_0 \) are finite. Next, by (1.4), we have the inequality
\[
V_\alpha(x) \leq E \int_0^\infty e^{-\alpha t}[C(X_\alpha(s))ds + d|A|] \leq \frac{M_1}{\alpha} + E \int_0^\infty e^{-\alpha t}d|A|(t).
\]
Hence, using integration by parts and the estimate for \( E|A|(T) \), we obtain
\[
V_\alpha(x) \leq \frac{M_1}{\alpha} + \alpha \int_0^\infty e^{-\alpha t}(\gamma t + 2M_1)dt \leq \frac{M_1}{\alpha} + \frac{3}{\alpha} + 2M_1.
\]
Consequently, \( V_\alpha(x) \) is also finite and the following uniform estimate holds:
\[
\sup_{0<\alpha<\alpha_0} \sup_{[u,b]} \alpha V_\alpha(x) \leq M_0 \equiv (M_1 + \gamma + 2M_1\delta_0).
\]
Hence, part (i) follows.

For any admissible process $X_x$, introduce the cost functional

$$J(x, X_x, T) = E \int_0^T [C(X_x(t))dt + d|A(t)|].$$

For a given $\epsilon > 0$ and any $T > 0$, we pick a process $X_x$ so that $V_0(x, T) + \epsilon > J(x, X_x, T)$. Then for any $y$, consider the process $\tilde{X}_y(0-)=y$ and with an initial jump to the point $x$ so that $\tilde{X}_y(0)=x$. Thereafter it satisfies $\tilde{X}_y(t) \equiv X_x(t)$ for all $t > 0$. Hence we observe that $J(y, \tilde{X}_y, T) = |x-y| + J(x, X_x, T)$ and consequently, $V_0(y, T) < |x-y| + J(x, X_x, T) < |x-y| + V_0(x, T) + \epsilon$. Since $\epsilon$ is arbitrary and $x$ and $y$ are arbitrary points, we obtain

$$|V_0(x, T) - V_0(y, T)| \leq |x-y|.$$  \hspace{1cm} (2.7)

By the Theorems 4.3 and 5.5 of [28], $V_0$ satisfies $|V_0'(x)| \leq 1$ for all $x$ and hence $|V_0(x) - V_0(y)| \leq |x-y|$. Thus, part (ii) follows.

To prove part (iii), we can alter an argument from classical analysis (see p.107, [23]). We pick any constant $K_1$ so that $K_1 > \lambda_0$. Then, there is an admissible process $X_x(.)$ so that

$$\limsup_{T \to \infty} \frac{J(x, X_x, T)}{T} < K_1.$$  \hspace{1cm} (2.8)

Introduce a Borel measure $\nu$ on $[0, \infty)$, induced by the distribution function $F$, where $F$ defined by $F(T) = J(x, X_x, T)$ for all $T > 0$. Hence, $\nu([0, T]) = F(T)$ for all $T > 0$. Also, introduce the function $G$ on $[0, \infty)$, by $G(T) = \frac{F(T)}{T}$. Therefore,

$$\alpha \int_0^\infty e^{-\alpha t}d\nu(t) = \alpha^2 \int_0^\infty e^{-\alpha t}F(t)dt \quad \text{(by integration by parts)}$$

$$= \alpha^2 \int_0^\infty e^{-\alpha t}(t+1)G(t)dt = \int_0^\infty e^{-y}(y+\alpha)G\left(\frac{y}{\alpha}\right)dy.$$  \hspace{1cm} (2.9)

By (2.8), $G(t) < K_1$ for all $t > T_0 > 1$ and $\|G\|_\infty = \sup_{[0, \infty)} |G(t)| < \infty$. Hence, by (2.9),

$$\alpha V_0(x) = \alpha \int_0^\infty e^{-\alpha t}d\nu(t)$$

$$= \int_0^{\alpha T_0} e^{-y}(y+\alpha)G\left(\frac{y}{\alpha}\right)dy + \int_{\alpha T_0}^\infty e^{-y}(y+\alpha)G\left(\frac{y}{\alpha}\right)dy$$

$$< \alpha^2(T_0^2 + T_0)\|G\|_\infty + K_1 \int_0^\infty e^{-y}(y+\alpha)dy$$

$$< \alpha^2(T_0^2 + T_0)\|G\|_\infty + K_1(1 + \alpha).$$
Consequently, \( \limsup_{\alpha \to 0} \alpha V_{\alpha} (x) \leq K_1 \). Since \( K_1 > \lambda_0 \) is arbitrary this implies that \( \limsup_{\alpha \to 0} \alpha V_{\alpha} (x) \leq \lambda_0 \). Next \( V_0 (x, T) \leq F(T) \) for all \( T > 0 \) and by (2.8), we obtain \( \limsup_{T \to \infty} \frac{V_0 (x; T)}{F(T)} \leq \limsup_{T \to \infty} \frac{F(T)}{F(T)} < K_1 \). But \( K_1 > \lambda_0 \) is arbitrary. Hence \( \limsup_{T \to \infty} \frac{V_0 (x; T)}{F(T)} \leq \lambda_0 \). Thus, the proof of part (iii) is complete.

3 Optimality of the zero control

First we introduce the state process \( Z_x (\cdot) \) which corresponds to zero control policy, namely, \( A(t) \equiv 0 \) for all \( t \) in (1.1). Let \( Z_x (\cdot) \) be a weak solution ([14]) to

\[
Z_x (t) = x + \int_0^t \mu (Z_x (s)) ds + \int_0^t \sigma (Z_x (s)) dW(s) \tag{3.1}
\]

where \( W(t) \) is a one dimensional Brownian motion. The existence of \( Z_x (t) \) for all \( t \geq 0 \) and the finiteness of the first moment \( E|Z_x (t)| \) for each \( t \geq 0 \) are obtained in section 4 of [28] (see also Chapter 5, Theorem 5.15 in [14]). The main theorem in this section is the following:

**Theorem 3.1** Assume (1.6), (1.7), (1.8) and that \( H(x) \leq 0 \) for all \( x \), where \( H \) is given in (1.9). Then the following hold:

(i) \( \lim_{\alpha \to 0} \alpha V_{\alpha} (x) = \Lambda_0 \) exists where \( \Lambda_0 \) is a constant. Moreover, this limit converges uniformly over compact sets.

(ii) The process \( Z_x (\cdot) \) of (3.1) is an optimal process which corresponds to the zero control policy for the ergodic control problem in (1.3). Its value \( \lambda_0 = \Lambda_0 \), where \( \Lambda_0 \) is the limit in part (i).

This theorem implies that, under the above set of assumptions, there is no optimal target band for the exchange rates related to the question 1 of our example on foreign exchange rates in section 1. Furthermore, an optimal policy for the central bank is not to intervene at all.

To prove this theorem, first we describe some results related to the value function \( V_{\alpha} (x) \), which were developed in sections 3 and 4 of [28]. Let \( Y(.) \) be the weak solution to

\[
Y(T) = x + \int_0^T [\sigma (Y(t)) \sigma' (Y(t)) + \mu (Y(t))] dt + \int_0^T \sigma (Y(t)) dB(t) \tag{3.2}
\]
where \( \{B(t) : t \geq 0\} \) is a Brownian motion. The existence and uniqueness of a weak solution to (3.2) follows from theorem 5.15 of chapter 5 in [14]. This process was also introduced in the equation (3.1) of [28]. Next we consider the function \( W_\infty \) introduced in Lemma 4.2 of [28] and relabel it as \( W_\alpha \) to specify its dependence on \( \alpha \). Then, by Lemma 4.2 of [28], \( W_\alpha (x) \) has the stochastic representation

\[
W_\alpha (x) = E_x \int_0^{\tau_\infty} e^{-\int_0^t (\alpha - \mu'(Y(s))) ds} C'(Y(t)) dt, \tag{3.3}
\]

where \( \tau_\infty \) is the explosion time of the \( Y(\cdot) \) process. Next, the assumption \( H(x) \leq 0 \) for all \( x \) implies that \( |C'(x)| < (\alpha - \mu'(x)) \) for all \( x \). Using this estimate in (3.3), we obtain \( |W_\alpha (x)| < 1 \) for all \( x \). Furthermore, as in Lemma 4.2 of [28], \( W_\alpha \) satisfies

\[
\frac{\sigma^2(x)}{2} W''_\alpha (x) + (\sigma(x)\sigma'(x) + \mu(x)) W'_\alpha (x) - (\alpha - \mu'(x)) W_\alpha (x) + C'(x) = 0 \tag{3.4}
\]

for all \( x \) and \( |W_\alpha (x)| < 1 \) for all \( x \).

Theorem 4.3 of [28] also implies the following representation for the value function \( V_\alpha \):

\[
V_\alpha (x) = \frac{\sigma^2(0)}{2\alpha} W'_\alpha (0) + \int_0^x W_\alpha (u) du. \tag{3.5}
\]

Next, we prove a technical lemma.

**Lemma 3.2** Let \( W_\alpha \) be as in (3.3) above. Then the following results hold.

(i) \( \lim_{\alpha \to 0} W_\alpha (x) \) exists for all \( x \). Let \( W_0 (x) \triangleq \lim_{\alpha \to 0} W_\alpha (x) \), then \( W_0 (x) \) has the stochastic representation

\[
W_0 (x) = E_x \int_0^{\tau_\infty} e^{-\int_0^t (\alpha - \mu'(Y(s))) ds} C'(Y(t)) dt. \tag{3.6}
\]

(ii) \( W_0 (\cdot) \) also satisfies

\[
\frac{\sigma^2(x)}{2} W''_0 (x) + (\sigma(x)\sigma'(x) + \mu(x)) W'_0 (x) + \mu'(x) W_0 (x) + C'(x) = 0 \tag{3.7}
\]

and \( |W_0 (x)| \leq 1 \) for all \( x \).

(iii)

\[
\lim_{\alpha \to 0} W'_\alpha (0) = W'_0 (0). \tag{3.8}
\]
Proof. Consider the representation (3.3) for $W_\alpha$. Since $H(x) \leq 0$ for all $x$, we obtain $|C'(Y(t))|e^{-\int_0^t (\alpha - \mu'(Y(s)))ds} \leq -\mu'(Y(t))e^{-\int_0^t (\alpha - \mu'(Y(s)))ds}$.

Notice that, $-\int_0^\infty e^{\int_0^t \mu'(Y(s))ds} \mu'(Y(t))dt = 1 - e^{\int_0^{\infty} \mu'(Y(s))ds} \leq 1$.

Now using (3.3) and the dominated convergence theorem, it follows that $\lim_{\alpha \to 0} W_\alpha(x) \equiv W_0(x)$ exists and $W_0$ has the representation (3.6). Hence, part (i) follows.

Next, we integrate (3.4) twice and obtain

$$\frac{\sigma^2(0)}{2} W'_\alpha(0) \int_0^x \frac{2}{\sigma^2(r)} dr = W_\alpha(x) - W_\alpha(0) + 2 \int_0^x \frac{\mu(r)}{\sigma^2(r)} W_\alpha(r) dr$$

$$+ 2 \int_0^x \frac{C(r)}{\sigma^2(r)} dr - 2\alpha \int_0^x \frac{W_\alpha(u)}{\sigma^2(u)} du dr.$$

Since $\lim_{\alpha \to 0} W_\alpha(x) \equiv W_0(x)$ exists and $|W_\alpha(x)| < 1$ for all $x$, we obtain that the right hand side of the above equation converges as $\alpha$ tends to zero and $|W_\alpha(x)| \leq 1$ for all $x$. Therefore, $\lim_{\alpha \to 0} W'_\alpha(0)$ exists and we label it $\beta$. Hence,

$$\frac{\sigma^2(0)}{2} \beta \int_0^x \frac{2}{\sigma^2(r)} dr = W_0(x) - W_0(0) + \int_0^x \frac{2\mu(r)}{\sigma^2(r)} W_0(r) dr + \int_0^x \frac{2C(r)}{\sigma^2(r)} dr.$$

By differentiating this equation, we obtain $W'_\alpha(0) = \beta$ and $W_0$ satisfies the differential equation (3.7). Hence, the proofs of parts (ii) and (iii) are complete.

Now we are ready to prove the theorem.

Proof of Theorem 3.1.

By (3.5), we can write

$$\alpha V_\alpha(x) = \Lambda_\alpha + \alpha \int_0^x W_\alpha(u) du$$

(3.9)

for all $x$, where

$$\Lambda_\alpha = \frac{\sigma^2(0)}{2} W'_\alpha(0).$$

(3.10)

Next, we let

$$\Lambda_0 = \frac{\sigma^2(0)}{2} W'_0(0).$$

(3.11)

Using part (iii) of lemma 3.2, we have $\lim_{\alpha \to 0} \Lambda_\alpha = \Lambda_0$. Since $|W_\alpha(x)| < 1$ for all $x$, using (3.9) we obtain $|\alpha V_\alpha(x) - \Lambda_0| \leq |\Lambda_\alpha - \Lambda_0| + \alpha |x|$. Then $\lim_{\alpha \to 0} \alpha V_\alpha(x) = \Lambda_0$ and the convergence is uniform on compact sets. Thus, the proof of part (i) is complete.
To prove part (ii), we consider the process $Z_x(\cdot)$ in (3.1) and first we show
\[
\limsup_{T \to \infty} \frac{1}{T} E \int_0^T C(Z_x(t))dt = \Lambda_0.
\]
Under the assumptions of this section, $Z_x(\cdot)$ process is also optimal for the discounted control problem (1.4) as shown in [28]. Hence,
\[
V_\alpha(x) = E \int_0^\infty e^{-\alpha t} C(Z_x(t))dt = \int_0^\infty e^{-\alpha t} E[C(Z_x(t))]dt.
\]
Now we can apply the classical Abelian limit theorem (p.117, [23]) to obtain
\[
\lim_{\alpha \to 0} \alpha V_\alpha(x) = \lim_{T \to \infty} \frac{1}{T} E \int_0^T C(Z_x(t))dt.
\]
This theorem also guarantees the existence of the ergodic limit in the right hand side of the above equation. Consequently, \( \lim_{T \to \infty} \frac{1}{T} E \int_0^T C(Z_x(t))dt = \Lambda_0 \).

By part (iii) of the proposition 2.1, we know that \( \Lambda_0 \leq \lambda_0 \), where \( \lambda_0 \) is the value of (1.3). Therefore, we can conclude $Z_x(\cdot)$ is an optimal process for (1.3) and
\[
\lim_{\alpha \to 0} \alpha V_\alpha(x) = \Lambda_0 = \lambda_0.
\]
This completes the proof.

Remarks.
1. It should be noted that, if the process $Z_x(\cdot)$ has a stationary distribution $\pi$, then the limit $\Lambda_0$ is equal to $\int C(y)\pi(dy)$.
2. Under the assumptions of Theorem 3.1, one can show that the process $Z_x(\cdot)$ is also optimal for the finite-horizon problem with the value function $V_0(x, T)$ in (1.5). Here we sketch the proof.

Let $Q(x, T) = E \int_0^T C(Z_x(t))dt$ be the pay-off from $Z_x(\cdot)$ in (3.1), which corresponds to zero control. Then $Q$ satisfies
\[
\frac{\sigma^2(x)}{2} Q_{xx} + \mu(x) Q_x + C(x) = Q_t
\]
for all $x$ and $t > 0$. Also, $Q(x, 0) = 0$. Now let $U(x, T) = Q_x(x, T)$, then $U$ satisfies
\[
\frac{\sigma^2(x)}{2} U_{xx} + (\sigma(x)\sigma'(x) + \mu(x)) U_x + \mu'(x) U + C'(x) = U_t
\]
for all $x$ and $t > 0$, and $U(x, 0) = 0$. By Itô’s lemma, (with the same notation as in (3.6)), $U$ has the stochastic representation
\[
U(x, T) = E \int_0^{T\wedge T} e^{\int_0^t \mu'(s)ds} C'(Y(t))dt.
\]
Thus, as in the proof of Theorem 3.1, $|U(x, T)| \leq 1$ since $H(x) \leq 0$. Now, for any given process $X_x(\cdot)$, by applying Itô’s lemma to $Q(X_x(t), T-t)$, we obtain
\[
Q(x, T) \leq E \int_0^T [C(X_x(s))ds + d|A|(s)].
\]
Hence, \( Q(x,T) = V_0(X,T) \) and \( Z_x(.) \) is optimal for the finite time horizon problem (1.5).

### 4 Optimality of a reflected diffusion

In this section, we assume the following conditions (4.1) and (4.2) in addition to the basic assumptions (1.6)-(1.8). Let the function \( H \) be as in (1.9). We assume the existence of a constant \( \alpha_0 > 0 \) which satisfies the following:

(i) For each \( 0 \leq \alpha < \alpha_0 \), there exist two points \( \theta_\alpha < 0 < \beta_\alpha \) so that \( H(x) > \alpha \) outside \([\theta_\alpha, \beta_\alpha]\). Furthermore, if \( \alpha > 0 \), then \( H(x) < \alpha \) in \((\theta_\alpha, \beta_\alpha)\). Finally, if \( \alpha = 0 \), then \( \{x: H(x) \leq 0\} = [\theta_0, \beta_0] \).

(ii) For each \( 0 < \alpha < \alpha_0 \), there are two constants \( \epsilon_\alpha > 0 \) and \( M_\alpha > 0 \) so that \( H(x) + \epsilon_\alpha \mu'(x) > (1 + \epsilon_\alpha) \alpha \) for all \( |x| > M_\alpha \).

**Remarks.**

1. Without loss of generality, we assume that \( \alpha_0 < \delta_0 \) where \( \delta_0 \) is given in (1.2).
2. By (4.1), \( H(0) \leq 0 \) and \( \theta_\alpha \leq \theta_0 < 0 < \beta_0 \leq \beta_\alpha \) for each \( 0 \leq \alpha < \alpha_0 \).
3. By (4.2), it follows that \( \lim_{x \to -\infty} C(x) + \mu(x) = +\infty \) and \( \lim_{x \to +\infty} C(x) - \mu(x) = +\infty \).
4. For each \( \alpha \) in \((0, \alpha_0)\), the above assumptions imply those of section 5 in [28] and hence we can use the results related to \( V_\alpha(x) \) in there.

Our main theorem in this section is the following:

**Theorem 4.1** Assume (4.1) and (4.2), in addition to the basic assumptions (1.6)-(1.8). Then the following hold:

(i) \( \lim_{\alpha \to 0} \sup_{|x| \leq K} |\alpha V_\alpha(x) - \lambda_0| = 0 \) for each \( K > 0 \).

(ii) There exist two points \( a^* \) and \( b^* \) so that the corresponding reflected diffusion process on the state space \([a^*, b^*] \) (if the initial point is outside this interval, then there will be an initial jump to the nearest point of the interval) is an optimal state process for the ergodic control problem (1.3).
Hence the optimal control policy is given by the difference of two local time processes at \( a^* \) and \( b^* \).

This theorem verifies the existence of an optimal target band for our example on exchange rates in section 1, under the above set of assumptions. In this case, the optimal intervention policy of the central bank involves local-time processes of the exchange rate as described in the above theorem.

First we gather the necessary technical results in Lemma 4.2.

**Lemma 4.2** Assume the same assumptions as in theorem 4.1. Let \( l_0 \) be any limit point of the set \( \{ \alpha V_\alpha(0) \} \) as \( \alpha \) tends to zero. Then, there exist two points, \( a^*, b^* \) and a continuously differentiable function \( W_0 \) defined on \( \mathbb{R} \) satisfying the following conditions:

(i) \( -\infty < a^* < \theta_0 < \beta_0 < b^* < +\infty \), where \( \theta_0 \) and \( \beta_0 \) are given in (4.1).

(ii) \( W_0 \) satisfies

\[
\sigma^2(x) W_0''(x) + \mu(x) W_0'(x) + C(x) = l_0 \text{ for } a^* < x < b^*.
\]

(iii) \( W_0(x) = -1 \) for all \( x \leq a^* \), \( W_0(x) = +1 \) for all \( x \geq b^* \) and \( |W_0(x)| \leq 1 \) for all \( x \).

(iv) The value of \( l_0 \) can be identified by the formula

\[
l_0 = \frac{e^{2b^*} \int_0^{b^*} \rho(u)du + e^{-2a^*} \int_0^{a^*} \rho(u)du}{2D} + \int_{a^*}^{b^*} C(u)\phi(u)du,
\]

where \( D = \int_{a^*}^{b^*} \frac{1}{\sigma^2(x)} e^{2\int_0^x \rho(u)du} dx \), \( \rho(x) = \frac{\mu(x)}{\sigma^2(x)} \) on \( [a^*, b^*] \) and the density function \( \phi \) is given by \( \phi(x) = \frac{1}{D} \frac{1}{\sigma^2(x)} e^{2\int_0^x \rho(u)du} \) on \( [a^*, b^*] \).

**Proof.** Using Proposition 5.4, equation (5.28) and Theorem 5.5 of [28], we obtain the following representation for the value function \( V_\alpha \):

\[
V_\alpha(x) = V_\alpha(0) + \int_{0}^{x} W_\alpha(u)du.
\]

Here, we write \( W_\alpha \) for the function \( W \) in the Proposition 5.4 of [28] to represent the dependence on \( \alpha \). We also observe that

\[
V_\alpha(x) = \frac{\sigma^2(0)}{2\alpha} W_\alpha(0) + \int_{0}^{x} W_\alpha(u)du.
\]

(4.3)

Furthermore, by Proposition 5.4 and Theorem 5.5 of [28], for each \( \alpha \), there exist two points \( a^*_\alpha, b^*_\alpha \) and a \( C^1 \) function \( W_\alpha \) so that \( a^*_\alpha < \theta_0 < \beta_0 < b^*_\alpha \) and \( W_\alpha \) satisfies

\[
\frac{\sigma^2(x)}{2} W_\alpha''(x) + (\sigma(x)\sigma'(x) + \mu(x)) W_\alpha'(x) - (\alpha - \mu'(x)) W_\alpha(x) + C'(x) = 0 \quad (4.4)
\]
for \( a^*_n < x < b^*_n \). Also,

\[
W_\alpha(a^*_n) = -1, \quad W_\alpha(b^*_n) = 1, \quad |W_\alpha(x)| < 1 \text{ on } (a^*_n, b^*_n), \tag{4.5}
\]

and

\[
W'_\alpha(a^*_n) = W'_\alpha(b^*_n) = 0, \quad W_\alpha(x) = -1 \text{ for } x \leq a^*_n \quad \text{and} \quad W_\alpha(x) = 1 \text{ for } x \geq b^*_n. \tag{4.6}
\]

This solution \( W_\alpha \) was obtained in [28] by deriving a solution to an optimal stopping problem. For details, we refer to [28]. Now consider the limit point \( l_0 \) of the set \( \{ \alpha V_\alpha(0) \} \) as \( \alpha \) tends to zero. Then, there is a decreasing sequence \( \{ \alpha_n \} \) so that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \lim_{n \to \infty} \alpha_n V_{\alpha_n}(0) = l_0 \). By the part (iii) of Proposition 2.1, it follows that \( l_0 \leq \lambda_0 \). Hence, there is a constant \( C_0 > 0 \) so that \( 0 < \alpha_n V_{\alpha_n}(0) < C_0 \) for all \( n \). Notice that \( [\theta_0, \beta_0] \subseteq [a^*_n, b^*_n] \). We intend to show that there is a finite \( K > 0 \) so that \( [a^*_n, b^*_n] \subseteq [-K, K] \) for all \( \alpha_n \). Next, by integrating (4.4) on \([0, b^*_n]\), we obtain

\[
C(b^*_n) + \mu(b^*_n) - \frac{\sigma^2(0)}{2} W'_{\alpha_n}(0) = \alpha_n \int_0^{b^*_n} W_{\alpha_n}(x) dx < \alpha_n b^*_n.
\]

Hence,

\[
\int_0^{b^*_n} [H(x) - \alpha_n] dx = C(b^*_n) + \mu(b^*_n) - \alpha_n b^*_n - \frac{\sigma^2(0)}{2} W'_{\alpha_n}(0) = \alpha_n V_{\alpha_n}(0) < C_0
\]

where \( C_0 \) is a constant independent of \( n \).

But, from (4.2), \( \int_0^{\infty} [H(x) - \alpha_n] dx = \infty \) for each \( n \). Hence, by picking \( n = 1 \), we have a constant \( K_1 > 0 \) so that \( \int_0^r [H(x) - \alpha_1] dx > C_0 \) for all \( x > K_1 \).

But, using (4.7) we obtain, \( \int_0^{b^*_n} [H(x) - \alpha_1] dx < \int_0^{b^*_n} [H(x) - \alpha_n] dx < C_0 \). Consequently,

\[
\beta_0 \leq b^*_n < K_1 \quad \text{for all } n. \tag{4.8}
\]

Similarly, by integrating (4.4) on \([a^*_n, 0]\), we obtain

\[
\int_{a^*_n}^0 [H(x) - \alpha_n] dx = C(a^*_n) - \mu(a^*_n) - \alpha_n |a^*_n| < \alpha_n V_{\alpha_n}(0) < C_0 \tag{4.9}
\]

for all \( n \). Then, using (4.2) and following an argument similar to above, we obtain a constant \( K_2 > 0 \) so that

\[
-K_2 < a^*_n \leq \theta_0 \quad \text{for all } n \tag{4.10}
\]
Now we let $K = \max\{K_1, K_2\}$, then by (4.8) and (4.10), we obtain

$$[a_n^*, b_n^*] \subseteq [-K, K] \quad \text{for all } n. \quad (4.11)$$

In the rest of the proof we show that \{W_{\alpha_n}\} and \{W'_{\alpha_n}\} are equicontinuous families and conclude that \{W_{\alpha_n}\} converges (possibly through a subsequence) to the desired function $W_0$ which satisfies parts (ii) and (iii) of the lemma. For this, we consider the sequence of functions $(W_{\alpha_n})$ defined on $[-K, K]$. By integrating (4.4), we have

$$\frac{\sigma^2(x)}{2}W_{\alpha_n}'(x) = \alpha_nV_{\alpha_n}(0) + \alpha_n \int_0^x W_{\alpha_n}(u)du - C(x) - \mu(x)W_{\alpha_n}(x)$$

Using the facts that $\inf_{[-K,K]} \sigma^2(x) > 0$, $|W_{\alpha_n}(x)| \leq 1$ for all $x$, the functions $\mu, \sigma$ and $C$ are bounded on $[-K, K]$ and $0 < \alpha_nV_{\alpha_n}(0) < C_0$ for all $n$, we obtain

$$\sup_{n} \sup_{[-K,K]} |W_{\alpha_n}'(x)| < C_1$$

where $C_1$ is a constant. (4.12)

Now, using (4.4) and (4.12) together with the same reasoning as above, we conclude

$$\sup_{n} \sup_{[a_n^*, b_n^*]} |W_{\alpha_n}''(x)| < C_2 \quad (4.13)$$

where $C_2 > 0$ is a constant. Here, at the end points $a_n^*$ and $b_n^*$, we considered the one sided limits $W_{\alpha_n}''(a_n^*+)$ and $W_{\alpha_n}''(b_n^-)$. Next, using (4.12), (4.13) and the fact that $W_{\alpha_n}''(x) = 0$ outside $[a_n^*, b_n^*]$, we conclude that \{W_{\alpha_n}\} and \{W'_{\alpha_n}\} are equicontinuous families on $[-K, K]$. By (4.11), we can pick a subsequence of $(\alpha_n)$, so that $(a_n^*)$ and $(b_n^*)$ converge to limit points $a^*$ and $b^*$ respectively. Furthermore, these limit points are inside $[-K, K]$. Since, \{W_{\alpha_n}\} and \{W'_{\alpha_n}\} are equicontinuous families on $[-K, K]$ (through a further subsequence, if necessary), using Arzela-Ascoli theorem, we can conclude that there exists a continuously differentiable function $W_0$ on the interval $[-K, K]$ which satisfies the following:

$$\lim_{\alpha_n \to 0} W_{\alpha_n}(x) = W_0(x) \quad \text{and} \quad \lim_{\alpha_n \to 0} W_{\alpha_n}'(x) = W_0'(x).$$

Furthermore, $-K \leq a^* \leq \theta_0 < \beta_0 \leq b^* \leq K$. Now by integrating (4.4) and letting $\alpha_n$ tend to zero in the resulting integral equation, we obtain that $W_0$ satisfies the differential equation in part (ii) of the lemma. To derive part (iii), we only need to extend $W_0$ to all $\mathbb{R}$, so that $W_0(x) = -1$ for $x \leq -K$ and
$W_0(x) = 1$ for $x \geq K$. Now, the equicontinuity of $\{W_\alpha\}$ and $\{W'_\alpha\}$, implies part (iii).

Next, as similar to the argument in (2.2) and (2.3), we can solve the first order differential equation in part (ii) with the boundary condition $W_0(a^*) = -1$ and then use the other boundary condition $W_0(b^*) = 1$ to obtain the formula for $l_0$. Hence, the proof of part (iv) is complete.

**Remarks.**

In the following proof of the theorem, we show that $l_0 = \lambda_0$. Hence, we obtain the uniqueness of $l_0$ as well as the existence of the limit $\lim_{\alpha \to 0} \alpha V_\alpha(0)$.

**Proof of Theorem 4.1.**

Introduce the function $Q$ by

$$Q(x) = \int_0^x W_0(u)du$$

where $W_0$ is as in the previous lemma. Let $a^*$ and $b^*$ be also as in the previous lemma. Next, consider the reflected diffusion process $X^*_x$ on the interval $[a^*, b^*]$, given by (4.14). This process is positive recurrent on the interval $[a^*, b^*]$ and its ergodic limit for the cost functional can be derived explicitly as given below (see also chapter II, sec.6 in [9]). In the following discussion, we simply assume $x$ is in $[a^*, b^*]$, since an initial jump to the set $\{a^*, b^*\}$ does not alter the cost functional in (1.3). Let

$$X^*_x(t) = x + \int_0^t \mu(X^*_x(s))ds + \int_0^t \sigma(X^*_x(s))ds + A^*(t)$$

(4.14)

where

$$A^*(t) = L_{a^*}(t) - L_{b^*}(t) \text{ and } |A^*(t)| = L_{a^*}(t) + L_{b^*}(t).$$

(4.15)

The processes $L_{a^*}$ and $L_{b^*}$ are local time processes of $X^*_x$ at the points $a^*$ and $b^*$ respectively. Clearly, $X^*_x$ is an admissible process for (1.3). We apply Itô’s lemma to $Q(X^*_x(T))$ and use lemma 4.2 to obtain

$$Q(X^*_x(T)) = Q(x) + l_0T - E \int_0^T [C(X^*_x(t))dt + d|A^*(t)|].$$

Consequently,

$$\lim_{T \to \infty} \frac{1}{T} E \int_0^T [C(X^*_x(t))dt + d|A^*(t)|] = l_0.$$ 

Therefore, $l_0 \geq \lambda_0$ and the value of $l_0$ is given in part (iv) of lemma 4.2. But, $l_0 \leq \lambda_0$ from the Proposition 2.1. Therefore, $l_0 = \lambda_0$ and the above $X^*_x$ process

19
is an optimal state process. Furthermore, every limit point of \( \{\alpha V_\alpha(0)\} \) is equal to \( \lambda_0 \). Consequently, \( \lim_{\alpha \to 0} \alpha V_\alpha(0) = \lambda_0 \). Using part (ii) of the Proposition 2.1, we obtain \( \lim_{\alpha \to 0} \sup_{|x| \leq K} |\alpha V_\alpha(x) - \lambda_0| = 0 \) for each \( K > 0 \). Hence, the proofs of both parts (i) and (ii) of the Theorem 4.1 are complete.

5 Asymptotics for \( V_0(x, T) \)

In this section, we intend to prove the following theorem which describes the long term behavior of \( V_0(x, T) \) defined in (1.5).

**Theorem 5.1** Under the assumptions of Theorem 3.1 or Theorem 4.1, the following Abelian limit holds:

For each \( K > 0 \),

\[
\lim_{T \to \infty} \sup_{|x| \leq K} \left| \frac{V_0(x, T)}{T} - \lambda_0 \right| = 0.
\]  

**Proof.** It suffices to show \( \liminf_{T \to \infty} \frac{V_0(0, T)}{T} \geq \lambda_0 \), because this together with parts (ii) and (iii) of Proposition 2.1 implies (5.1). First, we observe that a given process \( X_x(\cdot) \) which satisfies (1.1) on \([0, T]\) with the condition \( E \int_0^T [C(X_x(t))dt + d|A|(t)] < \infty \) can be extended to \([0, \infty)\) as an admissible process for (1.3), simply by using the zero control policy on \([T, \infty)\). Hence

\[
X_x(T + s) = X_x(T) + \int_T^{T+s} \mu(X_x(u))du + \int_T^{T+s} \sigma(X_x(u))dW(u)
\]

where \( \{W(t) : t \geq 0\} \) is a Brownian motion. Since we have observed that the process corresponding to zero control policy is an admissible process in section 3, it easily follows that \( X_x(\cdot) \) is also an admissible process.

Second, we have observed that \( \limsup_{T \to \infty} \frac{V_0(0, T)}{T} \leq \lambda_0 \) in the Proposition 2.1. Thus, if we take a constant \( M_1 > \lambda_0 \), then there is a \( T_0 > 0 \) so that

\[
V_0(0, T) < M_1 T
\]  

for all \( T > T_0 \). Because of this fact, it suffices to consider the collection of admissible processes \( X_0(\cdot) \) which satisfy (1.1) together with

\[
E \int_0^T [C(X_0(t))dt + d|A|(t)] < M_1 T.
\]
Next, we establish that the quantity $\frac{E|X_0(T)|}{T}$ is bounded for such an admissible process $X_0(\cdot)$. For this consider the even $C^2$ function $\phi$ defined by

$$
\phi(x) = \frac{1}{8}(3 + 6x^2 - x^4)I_{|x|<1} + |x|I_{|x|\geq1}.
$$

Here, $I_A$ denotes the indicator function of the set $A$. Then $0 \leq \phi'(x) \leq 1$ for $x \geq 0$ and $1 + \phi(x) \geq |x|$ for all $x$. Also, $\phi$ is non-negative and $\phi'(x)\mu(x) \leq 0$ for all $x$. Now, we use the generalized Itô’s lemma ([18], p.285) together with the above facts to obtain

$$
E\phi(X_0(T)) \leq E \int_0^T \frac{\sigma^2(X_0(s))}{2} \phi''(X_0(s))I_{[-1,1]}(X_0(s))ds + E|A|(T).
$$

Since we can find a constant $C_2 > 0$ so that $\sup_{[-1,1]}|\sigma^2(x)|\phi''(x)| \leq C_2$ and by (5.3), for each $T > T_0$ we obtain

$$
E|X_0(T)| \leq 1 + E\phi(X_0(T)) \leq C_2T + E|A|(T) + 1 < (M_1 + C_2 + 1)T. \quad (5.4)
$$

Notice that the constants $M_1$ and $C_2$ are independent of the process $X_0(\cdot)$. Next, by the dynamic programming principle (or applying the generalized Itô’s lemma), we derive

$$
\alpha V_\alpha(0) \leq \alpha E \int_0^{T \wedge \tau_n} e^{-\alpha s}[C(X_0(s))ds + d|A|(s)]
$$

$$
+ E[e^{-\alpha(T \wedge \tau_n)}\alpha V_\alpha(X_0(T \wedge \tau_n))]. \quad (5.5)
$$

Here $(\tau_n)$ is a sequence of stopping times which satisfy the assumption (1.2). From the proofs of Theorems 3.1 and 4.1 (see the equations (3.9), (3.10) and (4.3)), $V_\alpha(x)$ has the representation

$$
\alpha V_\alpha(x) = \Lambda_\alpha + \alpha \int_0^x W_\alpha(u)du, \quad (5.6)
$$

where $\lim_{\alpha \to 0} \Lambda_\alpha = \lambda_0$ and $|W_\alpha(x)| \leq 1$ for all $x$. Hence, $\alpha V_\alpha(x) \leq \Lambda_\alpha + \alpha |x|$ for all $x$. Combining this with (5.5), we obtain

$$
\Lambda_\alpha[1 - E(e^{-\alpha(T \wedge \tau_n)})] \leq \alpha E \int_0^T [C(X_0(s))ds + d|A|(s)]
$$

$$
+ \alpha E[X_0(T \wedge \tau_n)]e^{-\alpha(T \wedge \tau_n)}. \quad (5.7)
$$

Notice that $E[X_0(T \wedge \tau_n)]e^{-\alpha(T \wedge \tau_n)} \leq E[|X_0(\tau_n)|]e^{-\alpha \tau_n I_{[\tau_n<T]}} + E[|X_0(T)|e^{-\alpha T}]$ Now keeping $\alpha > 0$ fixed and letting $(\tau_n)$ tend to infinity and using (1.2), we
We have used (5.4) in the last inequality and here $K = (M_1 + C_2 + 1)$, where the constants $M_1$ and $C_2$ are as in (5.4). Thus, $K$ is independent of $\alpha$ as well as the process $X_0(\cdot)$. Therefore, $\Lambda_\alpha [1 - e^{-\alpha T}] \leq \alpha V_0(0, T) + \alpha K e^{-\alpha T}$. Consequently,

$$\Lambda_\alpha \frac{1 - e^{-\alpha T}}{\alpha T} \leq \frac{V_0(0, T)}{T} + \frac{K}{T} e^{-\alpha T}.$$ 

We choose $\alpha = \frac{\delta}{T}$ where $0 < \delta < 1$ as in the argument in section (vi) of [3] for large $T > 0$. Thus,

$$\Lambda_\alpha \frac{1 - e^{-\delta}}{\delta} \leq \frac{V_0(0, T)}{T} + \frac{K}{T} e^{-\delta}.$$ 

Since $\lim_{\alpha \to 0} \Lambda_\alpha = \lambda_0$, first we let $\delta$ tend to zero and then let $T$ tend to infinity to obtain $\lambda_0 \leq \liminf_{T \to \infty} \frac{V_0(0, T)}{T}$ as desired. This completes the proof.

### 6 A Constrained Optimization Problem

In this section, we address a constrained optimization problem which can be solved by using our results in section 4. For the purposes of this section, we need to strengthen the assumption (1.7) by the following assumption:

$$\int_{-\infty}^0 \frac{1 + \mu(x)}{\sigma^2(x)} dx = \int_0^{\infty} \frac{1 - \mu(x)}{\sigma^2(x)} dx = \infty. \quad (6.1)$$

At the end of this section, in Theorem 6.4, we will describe the results we can obtain when we replace (6.1) by the assumption (1.7). We will introduce further assumptions on $\mu$, $\sigma$ and $C$ after we describe the constrained optimization problem.

Consider the collection $\mathcal{U}$ of all admissible control systems used in the ergodic control problem (1.3). Let $\mathcal{U}_0$ be the subcollection of $\mathcal{U}$ where $\mu$ and $\sigma$ satisfy (1.6) and (6.1). Let $m > 0$ be any fixed positive real number. Here, we
address the following constrained minimization problem:

\[
\text{Minimize } \limsup_{T \to \infty} \frac{1}{T} E \int_0^T C(X_x(s)) ds \tag{6.2}
\]

subject to

\[
\limsup_{T \to \infty} \frac{E|A(T)|}{T} \leq m \tag{6.3}
\]

over all admissible systems in $U_0$. Notice that an initial jump does not affect our cost criteria or the constraint. Therefore, throughout this section, we simply consider the initial point $x$ to be the origin and omit the dependence on $x$ in our notation. To be more precise, for each $m > 0$, we define

\[
U_m = \{((\Omega, \mathcal{F}, P), (\mathcal{F}_t), W, A, X) \in U_0 : \limsup_{T \to \infty} \frac{E|A(T)|}{T} \leq m\}. \tag{6.4}
\]

The collection $U_m$ is non-empty, since the zero control policy developed in section 3 is there. The constrained minimization problem is to find the value function and characterize an optimal policy for

\[
\inf_{U_m} \limsup_{T \to \infty} \frac{1}{T} E \int_0^T C(X(s)) ds. \tag{6.5}
\]

We intend to characterize an optimal strategy that achieves the infimum in (6.5). We develop a “Lagrange multiplier” type method by introducing an unconstrained optimization problem whose cost criteria includes a “penalty rate” $p > 0$. This penalty rate can be considered as the Lagrange multiplier variable. For each $p > 0$, we can obtain an optimal strategy for the unconstrained problem from the results in section 4. Furthermore, we also obtain an explicit formula for the derivative of the value with respect to $p$. Then we show that there exist a unique value for $p$, say $p^*$, where the corresponding optimal control $A^*$ of the unconstrained problem satisfies $\lim_{T \to \infty} \frac{E|A^*(T)|}{T} = m$. This enables us to conclude that the same control policy is also optimal for the constrained minimization problem (6.5). At the end of this section, we also point out that if we assume (1.7) instead of (6.1), we can solve the constrained minimization problem with the same optimal policy only when $m > \gamma_0$, where $\gamma_0$ is a constant explicitly given in Theorem 6.4.

In continuous-time setting, the idea of using both Lagrange multipliers and Kuhn-Tucker characterization of optimal policies for constrained stochastic control problems is considered in [6]. A problem of finite-fuel singular control with dynamic constraints for the control process is addressed in [5]. The Lagrange
multiplier method was applied to a stochastic control problem with terminal conditions in page 241 of [22]. When the state space is a finite interval, a similar constrained optimization problem and an application to dynamic power control in wireless communication are addressed in [4] and we are motivated by their work.

Let \( p > 0 \) be a positive constant which represents the penalty rate. For each \( p > 0 \), we let

\[
\Gamma(p) = \inf_{U_0} \limsup_{T \to \infty} \frac{1}{T} E \left[ \int_0^T C(X(s)) ds + p \cdot |A|(T) \right].
\]  

(6.6)

Introduce the function \( C_p : \mathbb{R} \to [0, \infty) \) by

\[
C_p(x) = \frac{C(x)}{p} \quad \text{for all } x.
\]  

(6.7)

Notice that

\[
\frac{\Gamma(p)}{p} = \inf_{U_0} \limsup_{T \to \infty} \frac{1}{T} E \left[ \int_0^T \frac{C(X(s))}{p} ds + |A|(T) \right]
\]  

(6.8)

and the function \( C_p \) satisfies the assumption (1.8). Similar to the definition of \( H \) in (1.9), for each \( p > 0 \), we define the function \( H_p : \mathbb{R} \to \mathbb{R} \) by

\[
H_p(x) = \mu'(x) + \frac{|C'(x)|}{p}.
\]  

(6.9)

We make the following additional assumptions throughout this section. They will enable us to use the results in section 4. For each \( p > 0 \), we assume there exists a constant \( \alpha_0(p) > 0 \) which satisfies the conditions below.

(i) For each \( 0 \leq \alpha < \alpha_0(p) \), there exist two points \( \theta_\alpha(p) < 0 < \beta_\alpha(p) \) so that \( H_p(x) > \alpha \) outside the interval \([\theta_\alpha(p), \beta_\alpha(p)]\). Furthermore, if \( \alpha > 0 \) then \( H_p(x) < \alpha \) in \((\theta_\alpha(p), \beta_\alpha(p))\). Finally, if \( \alpha = 0 \) then \( \{ x : H_p(x) \leq 0 \} = [\theta_0(p), \beta_0(p)] \).

(6.10)

(ii) For each \( p > 0 \),

\[
\lim_{|x| \to \infty} \frac{\alpha_0(p) - \mu(x)}{|C'(x)|} = 0.
\]  

(6.11)

(iii) \( C(x) > 0 \) for all \( x \neq 0 \).

(6.12)

**Remarks.** The assumption (6.10) is similar to (4.1) in section 4 and (6.11)
clearly implies (4.2). The condition (6.12) enables us to simplify the proofs.

Examples.
The following examples of $\mu$, $\sigma$ and $C$ satisfy all the assumptions in this section:

(i) Let $\mu(x) = -\theta x^3$ for some $\theta > 0$, $\sigma(x) = 1 + x^2$ and $C(x) = x^{2n}$ for any $n \geq 2$.

(ii) Let $\mu(x) = -\theta x$ for some $\theta > 0$, $\sigma(x) = \sigma_0$ where $\sigma_0 > 0$ is a constant. Let $C(\cdot)$ be any $C^2$ convex function which has a unique minimum at zero, $C(0) = 0$ and $\lim_{|x| \to \infty} |C'(x)| = \infty$.

When $\mu$, $\sigma$ and $C$ satisfy all the assumptions in this section, then for each $p > 0$, $\mu$, $\sigma$ and $C_p$ clearly satisfy all the assumptions of Theorem 4.1. Therefore, for each $p > 0$, $\Gamma(\cdot)$ is finite and there is a finite interval $[a^*(p), b^*(p)]$ so that the corresponding reflecting diffusion process $X^*_p(\cdot)$ which satisfies (2.1) on this interval is an optimal process. The corresponding optimal bounded variation control process $A^*_p(\cdot)$ satisfies

$$A^*_p(t) = L_{a^*(p)}(t) - L_{b^*(p)}(t) \quad \text{for all } t > 0 \quad (6.13)$$

where $L_{a^*(p)}(\cdot)$ and $L_{b^*(p)}(\cdot)$ are local time processes of $X^*_p(\cdot)$ at the end points $a^*(p)$ and $b^*(p)$ respectively. By Lemma 4.2, we know that $a^*(p)$ and $b^*(p)$ satisfy

$$-\infty < a^*(p) \leq \theta_0(p) < 0 < \beta_0(p) \leq b^*(p) < \infty. \quad (6.14)$$

It is known that (see [9]), the reflected diffusion $X^*_p(\cdot)$ on the finite interval $[a^*(p), b^*(p)]$ has a unique stationary probability distribution with the probability density $\phi$ given below in (6.16). Therefore,

$$\lim_{T \to \infty} {1 \over T} E \int_0^T C(X^*_p(s))ds = \int_{a^*(p)}^{b^*(p)} C(u)\phi(u)du. \quad (6.15)$$

The density function $\phi$ is given by

$$\phi(x) = {1 \over D} {1 \over \sigma^2(x)} e^{2\int_0^x \rho(u)du}, \quad (6.16)$$

where $\rho(x) = {\mu(x) \over \sigma^2(x)}$ on the interval $[a^*(p), b^*(p)]$ and the normalization constant $D > 0$ is given by

$$D = \int_{a^*(p)}^{b^*(p)} {1 \over \sigma^2(x)} e^{2\int_0^x \rho(u)du}dx. \quad (6.17)$$
Also, we can use the limit in (2.5) for (6.13) to obtain
\[
\lim_{T \to \infty} E|A_p^*(T)| \frac{E}{T} = \frac{e^{\int_0^\alpha \rho(u)du} + e^{-\int_0^\alpha \rho(u)du}}{2D}
\]
(6.18)
where the constant \( D > 0 \) is given in (6.17).

Consequently, \( \Gamma(p) \) has the representation
\[
\Gamma(p) = \lim_{T \to \infty} \frac{1}{T} E \int_0^T C(X_p^*(s))ds + p \cdot \lim_{T \to \infty} E|A_p^*(T)| \frac{E}{T}.
\]
(6.19)

Our next lemma shows the differentiability of \( \Gamma(p) \) and computes the derivative explicitly.

**Lemma 6.1** For each \( p > 0 \), consider the value function \( \Gamma(p) \) defined in (6.6).
Then the following statements are true.

(i) \( \Gamma(\cdot) \) is a differentiable, strictly increasing function and its derivative is given by
\[
\Gamma'(p) = \lim_{T \to \infty} \frac{E|A_p^*(T)|}{T}
\]
where \( A_p^*(\cdot) \) is the optimal control process described in (6.13).

(ii) \( \Gamma(\cdot) \) satisfies \( 0 < p \cdot \Gamma'(p) < \Gamma(p) \) for each \( p > 0 \) and \( \lim_{p \to 0} \Gamma(p) = 0 \).

(iii) The function \( \frac{\Gamma(p)}{p} \) is strictly decreasing on \((0, \infty)\).

**Proof.**
We introduce the function \( F(a, b) \) on the set \( \{(a, b) \in \mathbb{R}^2 : a < 0 < b\} \) by
\[
F(a, b) = \frac{e^{\int_0^\alpha \rho(u)du} + e^{-\int_0^\alpha \rho(u)du}}{\int_a^b \frac{2}{\sigma^2(x)}e^{\int_0^\alpha \rho(u)du}dx}
\]
(6.21)
where \( \rho \) is as in (6.16). Notice that \( \rho > 0 \) on \((-\infty, 0)\) and \( \rho < 0 \) on \((0, \infty)\). Using this fact, a direct computation yields
\[
\frac{\partial F}{\partial a}(a, b) > 0 \quad \text{and} \quad \frac{\partial F}{\partial b}(a, b) < 0 \quad \text{when} \quad a < 0 < b.
\]
(6.22)

By (6.18), for each \( p > 0 \), we observe that
\[
F(a^*(p), b^*(p)) = \lim_{T \to \infty} \frac{E|A_p^*(T)|}{T},
\]
(6.23)
Next, consider $0 < p < q$. By (6.6), it is clear that $\Gamma(p) \leq \Gamma(q)$. Since the optimal strategy $(X^*_p, A^*_p)$ for $\Gamma(p)$ is also an admissible strategy for $\Gamma(q)$, using the definitions of $\Gamma(p)$, $\Gamma(q)$ and (6.23), we obtain

$$0 \leq \Gamma(q) - \Gamma(p) \leq (q - p) \cdot F(a^*(p), b^*(p)).$$

Similarly, $(X^*_q, A^*_q)$ is optimal for $\Gamma(q)$ and an admissible strategy for $\Gamma(p)$ and hence

$$0 < (q - p) \cdot F(a^*(q), b^*(q)) \leq \Gamma(q) - \Gamma(p).$$

Combining these two inequalities, we obtain

$$0 < F(a^*(q), b^*(q)) \leq \frac{\Gamma(q) - \Gamma(p)}{q - p} \leq F(a^*(p), b^*(p)) \quad \text{when } 0 < p < q. \quad (6.24)$$

Notice that (6.24) also implies that $F(a^*(p), b^*(p))$ is a decreasing function in the variable $p$. Now let $p_0 > 0$ be fixed. Let $\delta_1 > 0$ be such that $0 < \delta_1 < p_0$. Then by (6.24), for all $0 < |h| < \delta_1$, we have

$$F(a^*(p_0 + \delta_1), b^*(p_0 + \delta_1)) \leq \frac{\Gamma(p_0 + h) - \Gamma(p_0)}{h} \leq F(a^*(p_0 - \delta_1), b^*(p_0 - \delta_1)). \quad (6.25)$$

Clearly, (6.25) implies the continuity of $\Gamma(\cdot)$ at $p_0$. Moreover, it also shows that if $\lim_{p \to p_0} F(a^*(p), b^*(p)) = F(a^*(p_0), b^*(p_0))$ then $\Gamma(\cdot)$ is differentiable at $p_0$ and $\Gamma'(p_0) = F(a^*(p_0), b^*(p_0))$. Notice that, if $a^*(\cdot)$ and $b^*(\cdot)$ are continuous at $p = p_0$, then by (6.21), $\lim_{p \to p_0} F(a^*(p), b^*(p)) = F(a^*(p_0), b^*(p_0))$ holds. Therefore, to prove part(i) of the lemma, it suffices to show the continuity of the functions $a^*(\cdot)$ and $b^*(\cdot)$. Here, we prove the continuity of $b^*(\cdot)$. The proof of the continuity of $a^*(\cdot)$ is very similar and therefore, we omit it.

By the parts(ii) and (iii) of lemma 4.2 (notice that $a^*_0 = \lambda_0$ there, as shown in the proof of theorem 4.1), we have

$$\frac{C(b^*(p))}{p} + \mu(b^*(p)) = \frac{C(a^*(p))}{p} - \mu(a^*(p)) = \frac{\Gamma(p)}{p}. \quad (6.26)$$

First we show that $b^*(\cdot)$ is bounded on the interval $[p_0 - \delta_1, p_0 + \delta_0]$ and notice that $b^*(p) > 0$ by (6.14). By the monotonicity of $\Gamma(\cdot)$, (6.9) and by (6.26), we have

$$\frac{\Gamma(p_0 + \delta_1)}{p_0 - \delta_1} \geq \frac{\Gamma(p)}{p} = \int_0^{b^*(p)} H_p(u)du \geq \int_0^{b^*(p)} H_{p_0 + \delta_1}(u)du.$$

But, (6.11) implies that $\int_0^\infty H_{p_0 + \delta_1}(u)du = \infty$. Therefore, we can conclude that $b^*(\cdot)$ is a bounded function on $[p_0 - \delta_1, p_0 + \delta_1]$. Now consider a sequence
Consequently, we have \( \lim_{n \to \infty} b_n = p_0 \). Now let \( m_0 \) be any limit point of the bounded sequence \( (b_n) \). Then, \( m_0 \geq 0 \) by (6.14). By (6.26) and by the continuity of \( \Gamma(\cdot) \), we obtain \( \Gamma(m_0) = \Gamma(p_0) > 0 \) and consequently \( m_0 > 0 \). But \( b^*(p_0) > 0 \) and also satisfies \( \frac{C(b^*(p_0))}{p_0^2} + \mu(b^*(p_0)) = \frac{\Gamma(p_0)}{p_0} > 0 \). By (6.10), we have \( H_{p_0}(x) \leq 0 \) on \([0, \beta_0(p_0)]\) and \( H_{p_0}(x) > 0 \) on \((\beta_0(p_0), \infty)\). Therefore, \( \{x \geq 0 : \frac{C(x)}{p_0} + \mu(x) > 0\} \subseteq (\beta_0(p_0), \infty) \) and \( \frac{C(x)}{p_0} + \mu(x) \) is strictly increasing on \((\beta_0(p_0), \infty)\) and hence \( m_0 = b^*(p_0) \). This yields that \( b^*(\cdot) \) is continuous at \( p = p_0 \). A similar proof yields the continuity of \( a^*(\cdot) \) at \( p = p_0 \). Consequently, we have \( \lim_{p \to p_0} F(a^*(p), b^*(p)) = F(a^*(p_0), b^*(p_0)) \). This together with (6.25) implies that \( \Gamma(\cdot) \) is differentiable at \( p_0 \) and its derivative is given by \( \Gamma'(p_0) = F(a^*(p_0), b^*(p_0)) > 0 \). Hence, the proof of part (i) is complete.

To prove part (ii), using (6.15)-(6.19), (6.23) and the above result we can write

\[
\Gamma(p) = \int_{a^*(p)}^{b^*(p)} C(u)\phi(u)du + p \cdot \Gamma'(p)
\]

where \( \phi \) is given in (6.16). By (6.12) and (6.16), it is clear that the above integral is strictly positive. Therefore, \( \Gamma(p) > p \cdot \Gamma'(p) > 0 \) holds. To show \( \lim_{p \to 0} \Gamma(p) = 0 \), we consider the reflecting diffusion process described in (2.1) with \( a = -\sqrt{p} \) and \( b = \sqrt{p} \) on the interval \([-\sqrt{p}, \sqrt{p}]\) and obtain the following upper bound for \( \Gamma(p) \):

\[
0 < \Gamma(p) \leq \max\{C(\sqrt{p}), C(-\sqrt{p})\} + p \cdot F(-\sqrt{p}, \sqrt{p}).
\]

Clearly, \( \lim_{p \to 0^+} \max\{C(\sqrt{p}), C(-\sqrt{p})\} = 0 \). A direct computation shows that

\[
\lim_{p \to 0^+} \sqrt{p} \cdot F(-\sqrt{p}, \sqrt{p}) = \frac{\sigma^2(0)}{2}. \]

Therefore, \( \lim_{p \to 0^+} p \cdot F(-\sqrt{p}, \sqrt{p}) = 0 \) and consequently, \( \lim_{p \to 0^+} \Gamma(p) = 0 \). This completes the proof of part (ii).

For the proof of part (iii), consider \( 0 < p < q \). Then the optimal strategy \((X^*_p, A^*_p)\) for \( \Gamma(p) \) is also an admissible strategy for \( \Gamma(q) \) and therefore, by using (6.15)-(6.19), we obtain

\[
\frac{\Gamma(p)}{p} > \int_{a^*(p)}^{b^*(p)} \frac{C(u)}{q} \phi(u)du + F(a^*(p), b^*(p)) = \lim_{T \to \infty} \frac{1}{T} E\left[ \int_0^T C_q(X^*_p(t))dt + |A^*_p(T)| \right] \geq \frac{\Gamma(q)}{q}.
\]
This completes the proof of lemma.

In our next lemma, we derive the second order properties of the function \( \Gamma(\cdot) \). We show that \( \Gamma(\cdot) \) is a strictly concave function and its derivative \( \Gamma'(\cdot) \) takes all the values in the interval \((0, \infty)\).

**Lemma 6.2** The functions \( a^*(\cdot), b^*(\cdot) \) and \( \Gamma(\cdot) \) satisfy the following conditions:

(i) The functions \( a^*(\cdot) \) and \( b^*(\cdot) \) are differentiable and their derivatives satisfy \( \frac{da^*}{dp} < 0 \) and \( \frac{db^*}{dp} > 0 \).

(ii) \( \lim_{p \to 0^+} a^*(p) = \lim_{p \to 0^+} b^*(p) = 0 \), \( \lim_{p \to \infty} a^*(p) = -\infty \) and \( \lim_{p \to \infty} b^*(p) = \infty \).

(iii) \( \Gamma(\cdot) \) is a twice differentiable function which is strictly concave on \((0, \infty)\). Furthermore, \( \lim_{p \to 0^+} \Gamma'(p) = \infty \) and \( \lim_{p \to \infty} \Gamma'(p) = 0 \).

**Proof.**

Introduce the function

\[
U(p, x) = \frac{C(x)}{p} + \mu(x)
\]

for \( p > 0 \) and \( x > 0 \). Then \( U(p, b^*(p)) = \frac{\Gamma(p)}{p} \) by (6.26) and \( \frac{dU}{dx}(p, x) = H_p(x) \) by (6.9). We notice that \( b^*(p) > \beta_0(p) > 0 \) by (6.14) and by the argument below (6.26). Hence, \( \frac{dU}{dx}(p, b^*(p)) = H_p(b^*(p)) > 0 \). Therefore, we can use the implicit function theorem to conclude that \( b^*(\cdot) \) is differentiable at \( p \). Now, by differentiating \( \Gamma(p) \) with respect to \( p \) using (6.26), we obtain

\[
p \cdot H_p(b^*(p)) \cdot \frac{db^*}{dp}(p) + \mu(b^*(p)) = \Gamma'(p).
\]

By the previous lemma, \( \Gamma'(p) > 0 \) and \( \mu(b^*(p)) < 0 \) by (1.6). Also, \( H_p(b^*(p)) > 0 \) as we noted above. Hence, we conclude that \( \frac{db^*}{dp}(p) > 0 \). A similar proof yields \( \frac{da^*}{dp}(p) < 0 \). This completes the proof of part (i).

Consequently, \( b^*(\cdot) \) is a strictly increasing function on \((0, \infty)\) and thus the limit \( \lim_{p \to \infty} b^*(p) \) exists. Let \( \lim_{p \to \infty} b^*(p) = b_0 \). By (6.26), we obtain \( \Gamma(p) = C(b^*(p)) + p \cdot \mu(b^*(p)) \). Using lemma 6.1 and by letting \( p \) tend to zero, we can conclude that \( b_0 \geq 0 \) and \( C(b_0) = 0 \). By (6.12), it follows that \( b_0 = 0 \). A similar proof shows \( \lim_{p \to 0^+} a^*(p) = 0 \).

Next, we intend to show \( \lim_{p \to \infty} b^*(p) = \infty \). Since \( b^*(\cdot) \) is strictly increasing, we let \( b_\infty = \lim_{p \to \infty} b^*(p) \). If \( b_\infty \) is finite, using (1.6) and (1.8) we obtain

\[
0 < \Gamma(p) = C(b^*(p)) + p \cdot \mu(b^*(p)) < C(b_\infty) + p \cdot \mu(b^*(1))
\]

29
for all \( p > 1 \). If \( b_\infty \) is finite, then the right hand side of the above expression tends to \(-\infty\) as \( p \) tends to infinity and this is a contradiction. Hence \( b_\infty = \infty \). The proof of \( \lim_{p \to -\infty} a^*(p) = -\infty \) is similar and this completes the proof of part (ii).

To prove part (iii), using (6.23) and lemma 6.1, we obtain the representation

\[
\Gamma'(p) = F(a^*(p), b^*(p)) \quad \text{for } p > 0. \tag{6.28}
\]

Since \( F(\cdot, \cdot) \) is differentiable, using the proof of part (i) of this lemma, we have that \( \Gamma'(\cdot) \) is twice differentiable and its second derivative is given by

\[
\Gamma''(p) = \frac{\partial F}{\partial a}(a^*(p), b^*(p)) \cdot \frac{da^*}{dp}(p) + \frac{\partial F}{\partial b}(a^*(p), b^*(p)) \cdot \frac{db^*}{dp}(p).
\]

Now using (6.22) and the above part (i) of the lemma, it is evident that \( \Gamma''(p) < 0 \). Next, using (6.21), (6.28) and part (ii) of this lemma, we obtain

\[
\lim_{p \to 0^+} \Gamma'(p) = \infty.
\]

To compute \( \lim_{p \to \infty} \Gamma'(p) \), again we use (6.21), (6.28), the fact that \( F(a^*(p), b^*(p)) \) is a decreasing function in the variable \( p \) and the concavity of \( \Gamma(\cdot) \). Then we can conclude that

\[
\lim_{p \to \infty} \Gamma'(p) = F(-\infty, \infty) = \frac{e^2 \int_{-\infty}^{\infty} \rho(u) du + e^{-2} \int_{-\infty}^{0} \rho(u) du}{\int_{-\infty}^{\infty} \frac{1}{\sigma^2(r)} e^{2} \int_{0}^{\infty} \rho(u) du dx}. \tag{6.29}
\]

By (1.6), we obtain \( 0 < \int_{-\infty}^{0} \rho(u) du \leq \infty \) and \(-\infty \leq \int_{0}^{\infty} \rho(u) du < 0 \). If \( \int_{-\infty}^{0} \rho(u) du = \infty \) and \( \int_{0}^{\infty} \rho(u) du = -\infty \), then by (6.29), clearly \( \lim_{p \to \infty} \Gamma'(p) = 0 \).

Next consider the case \( \int_{0}^{\infty} \rho(u) du \) is convergent. Let, \( \int_{0}^{\infty} \rho(u) du = -L \) where \( L \) is a positive constant. Then, by (6.1), \( \int_{0}^{\infty} \frac{1}{\sigma^2(r)} dr = \infty \). Therefore, the numerator of the right hand side of (6.29) is less than 2 while the denominator is greater than or equal to \( 2e^{-2L} \int_{0}^{\infty} \frac{1}{\sigma^2(r)} dr \). Hence, the denominator is infinite and consequently, \( \lim_{p \to \infty} \Gamma'(p) = 0 \). If the integral \( \int_{-\infty}^{0} \rho(u) du \) is convergent, a similar proof shows \( \lim_{p \to \infty} \Gamma'(p) = 0 \). This completes the proof.

**Remarks.**

The assumption (6.1) is used only in the proof of \( \lim_{p \to \infty} \Gamma'(p) = 0 \). If we assume (1.7) instead of (6.1), our conclusion for the limit \( \lim_{p \to \infty} \Gamma'(p) \) will be given by the right hand side of (6.29).

Next, we present the main theorem of this section.
Theorem 6.3 Assume (1.6),(1.8),(6.1) and (6.10)-(6.12). Then for any positive constant m > 0, the constrained optimization problem (6.5) has an optimal strategy of the type described in part (ii) of the Theorem 4.1: Namely, there exist two points a∗ < 0 and b∗ > 0 so that the reflecting diffusion process described in (4.14) and (4.15) on the state space [a∗, b∗] is an optimal state process.

Proof. Let m > 0 be a constant. Then by lemma 6.2, there is a unique constant p∗ > 0 so that Γ′(p∗) = m. Consider the optimal strategy (X∗p, A∗p) for (6.6) with p = p∗. Then, using lemma 6.1, we can write

\[ \Gamma(p^*) = \lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T C(X^*_p(u))du \right] + p^* \cdot m. \quad (6.30) \]

Now consider any admissible control system in \( U_m \) where \( U_m \) is given in (6.4). Let \( X(\cdot) \) be the state process and \( A(\cdot) \) be the corresponding control process. Then \( (X, A) \) is also an admissible strategy for \( \Gamma(p^*) \) and the following inequalities hold:

\[
\begin{align*}
\limsup_{T \to \infty} \frac{1}{T} E \int_0^T C(X(u))du + p^* \cdot m & \geq \limsup_{T \to \infty} \frac{1}{T} E \left[ \int_0^T C(X(u))du \right] + p^* \cdot \limsup_{T \to \infty} \frac{E|A|(T)}{T} \\
& \geq \limsup_{T \to \infty} \frac{1}{T} E \left[ \int_0^T C(X(u))du + p^*|A|(T) \right] \\
& \geq \Gamma(p^*). \quad (6.31)
\end{align*}
\]

Therefore, by (6.30) and (6.31), we conclude that \( \limsup_{T \to \infty} \frac{1}{T} E \int_0^T C(X(u))du \geq \limsup_{T \to \infty} \frac{1}{T} E \int_0^T C(X^*_p(u))du \). Hence, \( (X^*_p, A^*_p) \) is an optimal policy for the constrained optimization problem (6.5). This completes the proof.

When we replace the assumption (6.1) by (1.7), we can obtain a partial solution to the constrained optimization problem. To describe it, first we introduce the non-negative constant \( \gamma_0 \) by \( \gamma_0 \equiv F(\infty, \infty) \). Here, \( F(\infty, \infty) \) is defined as in the right hand side of the equation (6.29). Then we have the following result:

Theorem 6.4 Assume (1.6)-(1.8) and (6.10)-(6.12). Let \( \gamma_0 \) be the constant defined above. Then for each \( m > \gamma_0 \), the conclusion of the theorem 6.3 holds.
Proof. By lemma 6.2, $\Gamma'(\cdot)$ is a strictly increasing function whose range is $(\gamma_0, \infty)$. Then for a given $m > \gamma_0$, there exists a unique $p^* > 0$ so that $\Gamma'(p^*) = m$. Now, the rest of the proof is identical to that of the above theorem 6.3.

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References


