You need to show all your work. Answers without justification worth zero points.

1. Compute the general solution to each of the following differential equations.
   (i) \((t^2 + 1)\frac{dy}{dt} + 2ty = 6t + 3t^2\)
   (ii) \((4x^2y + 2x + 3y^2)\frac{dy}{dx} = -4xy^2 - 2y\)

2. Find the solution to the initial value problem
   \[
   \frac{dy}{dx} = \frac{2y^2 + 2xy^2}{1 + y}, \quad y(0) = 1.
   \]

3. a) Find the largest interval around \(t = 1\) where you can guarantee a solution to the initial value problem (Do not compute the solution.)
   \((t^2 - 16)\frac{dy}{dt} + (t^2 + 16)y = \frac{\cos t}{t} \quad y(1) = 3.\)
   b) Consider the differential equation
      \[
      \frac{dy}{dt} = 6y(y - 4)
      \]
      (i) Find all the equilibrium solutions and indicate whether they are stable or unstable.
      (ii) Find the regions where the solution curves are increasing, decreasing, concave up and concave down.
      (iii) Draw the direction field and illustrate your answers to (i) and (ii).

4. A tank with a capacity of 500 liters originally contains 200 liters of water with 100 grams of dissolved salt. Water containing 1 gram of salt per liter flows into the tank at a rate of 3 liters per minute and the mixture is drained out at a rate of 3 liters per minute.
   (i) Let \(Q(t)\) be the amount of salt (in grams) in the tank at time \(t\). Write down a differential equation for \(Q(t)\). Make sure to state the initial condition.
   (ii) Compute \(Q(t)\) and find \(\lim_{t \to \infty} Q(t)\).
1. Find the solution to the initial value problem

\[
\frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} - 8y = 0, \quad y(0) = 2 \quad \text{and} \quad y'(0) = 38.
\]

2. (i) Compute a fundamental set of solutions for

\[
\frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 4y = 0.
\]

(ii) Find the general solution to \(\frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 4y = 8 + 7 \sin t\).

3. Solve the initial value problem

\[
\frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} + 10y = 0, \quad y(0) = 5 \quad \text{and} \quad y'(0) = 26.
\]

4. True-False question. State whether each of the following statement is true or false. Then make sure to verify your answer.

   (i) A particular solution to \(\frac{d^2 y}{dt^2} - \frac{dy}{dt} - 2y = 7e^{-t}\) is of the form \(y_p(t) = ce^{-t}\) where \(c\) is a constant. [If this statement is true, compute \(c\). If false, write down the correct form for \(y_p\).]

   (ii) A particular solution of \(\frac{d^2 y}{dt^2} + y = \sin t\) is of the form \(y_p(t) = c_1 \cos t + c_2 \sin t\). [If the statement is true, compute \(c_1\) and \(c_2\). If it is false, write down the correct form.]

   (iii) The two functions \(f_1(t) = 2 \cos^2 t\) and \(f_2(t) = 1 - \sin^2 t\) are linearly independent.
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You need to show all your work. Answers without justification worth zero points.

1. (i) Compute a fundamental set of solutions for \( \frac{d^4y}{dt^4} + 8 \frac{d^2y}{dt^2} - 9y = 0 \).

   (ii) Find the general solution to \( \frac{d^4y}{dt^4} + 8 \frac{d^2y}{dt^2} - 9y = 7 - 9t - 9t^2 \).

2. (a) Compute a fundamental set for the differential equation \( \frac{d^4y}{dt^4} + 9y = 0 \).

   (b) Determine whether the set \( \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\} \) is linearly independent or dependent. Give your reasons.

3. Find the solution to the initial value problem

   \[ \frac{dx}{dt} = Ax, \quad \text{and} \quad x(0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad \text{where} \quad A = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}. \]

4. Let \( A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 5 & 0 & 7 \end{bmatrix} \).

   (i) Find all the eigen values and corresponding eigen vectors.

   (ii) Find a fundamental set of solutions for \( \frac{dx}{dt} = Ax \)
You need to show all your work. Answers without justification worth zero points.

1. Compute the general solution to the following first-order differential equations:
   
   (a) \[ \frac{dy}{dx} = \frac{8x^2y}{(1 + x^4)(2 + y^2)} \]
   
   (b) \[ x \frac{dy}{dx} = 4 - 4y + \frac{1}{x^3} \]

2. (i) Find a fundamental set of solutions for \[ \frac{d^2y}{dt^4} - 8 \frac{dy}{dt} + 15y = 0. \]
   
   (ii) Compute the general solution to the non-homogeneous equation
   
   \[ \frac{d^2y}{dt^2} - 8 \frac{dy}{dt} + 15y = 30t^2 - 32t + 5 - 10e^{3t}. \]

3. a) Find an integration factor of the form \( \mu(x) \) for the first-order differential equation
   
   \( (x^2 + 2xy + 4)dy + (2x^3y + 2x^2y^2 + 10xy + y^2)dx = 0. \)

   b) Write down a fundamental set of solutions for \[ \frac{d^6y}{dt^6} - y = 0. \]

4. Find the solution to the initial value problem
   
   \[ \frac{dx}{dt} = \begin{bmatrix} 4 & -5 \\ 4 & -4 \end{bmatrix} x \quad \text{and} \quad x(0) = \begin{pmatrix} 10 \\ 2 \end{pmatrix}. \]

5. Let \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \). It is given that \( \lambda = -1, 1 \) and 2 are eigen values of \( A \).
   
   (i) Compute eigen vectors for the above described eigen values.

   (ii) Find a fundamental set of solutions for the system \( \frac{dx}{dt} = Ax. \)

6. True-False question. State whether each statement is true or false and then justify your answer.
   
   (i) the differential equation \( \frac{dy}{dt} = y(4 - y^2) \) has no stable equilibrium points.

   (ii) Let \( A \) be a \( 3 \times 3 \) matrix with eigen values \(-1, -3 \) and \(-5 \). Consider any solution \( x(t) \) of the differential equation \( \frac{dx}{dt} = Ax. \) Then \( \lim_{t \to \infty} x(t) = 0. \)

   (iii) The set of vectors \( \left\{ \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\} \) is linearly independent.