EXAM 1 PREPARATION

1. (i) Determine whether the points $P = (1, -2, 3), Q = (2, 1, 0),$ and $R = (4, 7, -6)$ lie on the same straight line.
(ii) Find the center and radius of the sphere $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0.$

2. (i) Let $\vec{u} = 2\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{v} = 3\vec{j} + 4\vec{k}$.
   a) Compute the unit vector along $\vec{v}$.
   b) Compute the projection vector of $\vec{u}$ along $\vec{v}$.
   (ii) Let $\vec{x} = a\vec{i} + b\vec{j} + 2\vec{k}$ where $a$ and $b$ are real numbers. If $\vec{x}$ is perpendicular to both vectors $2\vec{i} - 3\vec{j} - \vec{k}$ and $\vec{i} - 4\vec{j} + 2\vec{k}$, compute $a$ and $b$.

3. Find the volume of the parallelepiped having $\vec{u} = 3\vec{i} - 5\vec{j} + \vec{k}, \vec{v} = 2\vec{j} - 2\vec{k},$ and $\vec{w} = 3\vec{i} + \vec{j} + \vec{k}$ as adjacent edges.

4. A particle of mass $m$ is subjected to a constant force $\vec{F}$ is set in motion. Its position vector at time $t$ is given by $\vec{r}(t) = t\vec{i} + (t^2 - 4t)\vec{j} + (t^2 - 6t + 1)\vec{k}$.
   i) Compute the velocity and acceleration at time $t$.
   ii) Find the constant force $\vec{F}$. (Your answer may depend on $m$)
   iii) In what time period does the particle move downward?

5. i) Find the parametric equations of the straight line passing through the point $(1, 2, 1)$ and is parallel to the vector $\vec{u} = 3\vec{i} + 2\vec{j} + 5\vec{k}$.
   ii) Compute the equation of the plane perpendicular to the curve $\vec{r}(t) = t \sin t \vec{i} + \cos t \vec{j} + t \vec{k}$ at the point $t = \frac{\pi}{2}$. 
1. i) Compute $PQ$ and $QR$. $PQ = i + 3j - 3k$ and $QR = 2i + 6j - 6k$. Hence $QR = 2PQ$ and thus $PQ$ and $QR$ are parallel. Thus, $P, Q, R$ are on the same straight line.

ii) Notice that $x^2 + y^2 + z^2 - \frac{7}{2}x + 2y + \frac{1}{9} = 0$. By completing the square, we get $(x - \frac{1}{2})^2 + (y + 1)^2 + z^2 = 1$. Hence, the center is $(\frac{1}{2}, -1, 0)$ and the radius is 1.

2. i) Notice that $|\vec{v}|^2 = 9 + 16 = 25$. Thus $\vec{v} = 5$. Unit vector along $\vec{v}$ is given by $\hat{v} = \frac{1}{5}(3j + 4k)$.

b) $\text{Proj}_\vec{u}(\vec{u}) = (\frac{\vec{u} \cdot \hat{v}}{|\hat{v}|})\hat{v} = \frac{1}{5} \cdot \frac{1}{5}(3j + 4k) = \frac{1}{25}(3j + 4k)$

ii) $\vec{x}$ is perpendicular to $2i - 3j + k$. Hence $\vec{x} \cdot (2i - 3j + k) = 0$. We obtain $2a - 3b - 2 = 0$. Similarly, $\vec{x} \cdot (4j + 2k) = 0$ and thus $a - 4b + 4 = 0$. We can solve $2a - 3b = 2$ and $a - 4b = -4$ to obtain $a = 4$ and $b = 2$.

3. Volume $\equiv |\vec{u} \cdot (\vec{v} \times \vec{w})| \equiv |\vec{v} \cdot (\vec{u} \times \vec{w})| \equiv |\vec{w} \cdot (\vec{u} \times \vec{v})|$.

We compute $\vec{v} \times \vec{w} = 4i - 6j - 6k$. Then $\vec{u} \cdot (\vec{v} \times \vec{w}) = 12 + 30 - 6 = 36$

Volume = $|\vec{u} \cdot (\vec{v} \times \vec{w})| = 36$

4. Velocity at time $t$ is $\vec{r}'(t) = i + (2t - 4)j + (2t - 6)k$. Acceleration at time $t$ is $\vec{r}''(t) = 2j + 2k$.

Notice that the acceleration is a constant vector. By Newton’s law $\vec{F} = m \cdot \vec{r}''$. Hence $\vec{F} = 2m(j + k)$.

If the particle is moving downward, then the vertical component of velocity (coefficient of the $k$ vector) has to be negative. Therefore the particle moves downward when $2t - 6 < 0$. Thus, $0 \leq t < 3$.

5. i) Parametric equation $(x-1, y-2, z-1) = t(3, 2, 5)$ where $t$ is a parameter. Thus, $x = 1 + 3t$, $y = 2 + 2t$, and $z = 1 + 5t$.

ii) Notice $\vec{r}'(\frac{t}{2}) = (\frac{3}{2}, 0, \frac{5}{2})$ and $\vec{r}''(\frac{t}{2}) = i - j + k$. $\vec{r}''(\frac{t}{2})$ is normal to the plane. Equation to the plane $(x - \frac{3}{2}) - y + (z - \frac{5}{2}) = 0 \Rightarrow x - y + z = \pi$. 
