Moment Generating Functions

Note Title 11/10/2008

Definition: The K^{th} moment of a r.v. Y is $E(Y^K)$ for K integer.

$K=0 \quad E(Y^0) = 1$

$k=1 \quad E(Y^1) = E(Y) = \mu$

$k=2 \quad E(Y^2) = \sigma^2 + \mu^2$

MGF of r.v. Y

$$E(e^{tY}) = m_Y(t) = \begin{cases} \sum_y e^{ty} \cdot p(y) & \text{if discrete r.v. } Y \\ \int_{-\infty}^{\infty} e^{ty} \cdot f(y)\,dy & \text{if Cont. r.v. } Y \end{cases}$$

Connection to moments

$$\left.\frac{\partial^K E(e^{tY})}{\partial t^K}\right|_{t=0} = E(Y^K)$$

$$m_Y(t) = E\left(e^{tY}\right)$$

$$= E\left[1 + \frac{tY}{1!} + \frac{(tY)^2}{2!} + \frac{(tY)^3}{3!} + \frac{(tY)^4}{4!} + \cdots\right]$$

$$= 1 + tE(Y) + \frac{t^2}{2!}E(Y^2) + \frac{t^3}{3!}E(Y^3) + \frac{t^4}{4!}E(Y^4) + \cdots$$

$$\left.\frac{\partial}{\partial t}\right|_{t=0} = 0 + E(Y) + tE(Y^2) + \frac{t^2}{2!}E(Y^3) + \frac{t^3}{3!}E(Y^4) + \cdots$$

$$= E(Y)$$

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} = E(Y^2) + tE(Y^3) + \frac{t^2}{2!}E(Y^4) + \cdots$$

$$= E(Y^2).$$

Etc.

Finding moments with $m_Y(t)$

Exponential $\quad m_Y(t) = (1-\beta t)^{-1}$

$$E(Y) = \left.\frac{\partial m_Y(t)}{\partial t}\right|_{t=0}$$

$$\left.\frac{\partial m_Y(t)}{\partial t}\right|_{t=0} = -1\,(1-\beta t)^{-2} \cdot -\beta = \left.\beta(1-\beta t)^{-2}\right|_{t=0}$$

$$= \beta$$

$$\left.\frac{\partial^2 m_Y(t)}{\partial t^2}\right|_{t=0} = -2\beta(1-\beta t)^{-3} \cdot -\beta$$

$$= \left.2\beta^2(1-\beta t)^{-3}\right|_{t=0} = 2\beta^2$$

$$\left.\frac{\partial^3 m_Y(t)}{\partial t}\right|_{t=0} = \left.3!\,\beta^3(1-\beta t)^{-4}\right|_{t=0} = 3!\,\beta^3$$

MGF is another way to describe a distribution.

Application

Y is $N(\mu, \sigma^2)$ $\qquad m_Y(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$

Standardize Y, $\quad Z = \dfrac{Y-\mu}{\sigma} = \frac{1}{\sigma}Y - \frac{\mu}{\sigma}$

Let Y have MGF $m_Y(t)$ and $Z = aY + b$
then

$$m_Z(t) = e^{tb} \cdot m_Y(at)$$

$$m_Z(t) = e^{t\left(-\frac{\mu}{\sigma}\right)} m_Y\left(\frac{1}{\sigma}t\right)$$

$$= e^{t\left(-\frac{\mu}{\sigma}\right)} e^{\mu\frac{t}{\sigma} + \frac{1}{2}\left(\frac{t}{\sigma}\right)^2 \cdot \sigma^2}$$

$$= e^{\frac{1}{2}t^2} \qquad e^{\mu t + \frac{1}{2}t^2\sigma^2}$$

Is a mgf of a $N(0,1)$ r.v. therefore

Z is $N(0,1)$

$E(e^{tY})$ Y is uniform θ_1 & θ_2

$$\text{pdf} = f(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1} & \theta_1 \le y \le \theta_2 \\ 0 & \text{elsewhere} \end{cases}$$

$$E(e^{tY}) = \int_{\theta_1}^{\theta_2} e^{ty} \cdot \frac{1}{\theta_2 - \theta_1}\, dy$$

$$= \left(\frac{1}{\theta_2 - \theta_1}\right) \frac{e^{ty}}{t}\Bigg|_{\theta_1}^{\theta_2} = \frac{1}{t(\theta_2 - \theta_1)}\left[e^{t\theta_2} - e^{t\theta_1}\right]$$

$$= \boxed{\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}}$$

Standard Uniform $\theta_1 = 0 \quad \theta_2 = 1$

$$m(t) = \frac{e^t - 1}{t}$$

Uniqueness Property

Two r.v. X & Y

$m_X(t)$ & $m_Y(t)$

If $m_X(t) = m_Y(t)$ then X & Y must have the same p.d.f.

$\Rightarrow$ This is one method for determining the dist. of r.v.

$Y \sim N(\mu, \sigma^2) \qquad \left(\frac{Y-\mu}{\sigma}\right)^2$

$Y_1, Y_2, Y_3, \ldots, Y_n$ all with $N(\mu, \sigma^2)$

$$\sum_{i=1}^{n} Y_i$$