Discrete Random Variables

Note Title 9/22/2008

$$S = \{ \begin{matrix} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{matrix}$$

$P(\text{each outcome}) = \frac{1}{36}$

Apply a real-valued function to the values in S.

$g_1(S) = \text{sum}$

$g_2(S) = \text{largest (max)}$

The result is called a random variable.

Y = random variable
= sum of 2 dice

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y	2	3	4	5	6	7	8	9	10	11	12
$p(y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Y = maximum of 2 dice

y	1	2	3	4	5	6
$p(y)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Notation Y = random variable

y = possible values

$p(y) = P(Y = y) = p_Y(y)$

<u>sum</u> $P(Y=4) = p(4) = \frac{3}{36}$

Summary of Y

(1) table (2) graph (3) formula

probability distribution function pdf

Summaries of Distribution

mean of observations $\bar{y} = \dfrac{\sum_{i=1}^{n} y_i}{n}$

Std. dev. of observations $S = \sqrt{\dfrac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n-1}}$

Why $n-1$? $\sum_{i=1}^{n} (y_i - \bar{y})^2 \Longrightarrow \sum_{i=1}^{n} (y_i - \bar{y}) = 0$

sumtwodicesim

2	3	4	5	6	7	8	9	10	11	12
276	559	801	1108	1466	1644	1363	1089	839	565	290

$$\bar{y} = \frac{2 \cdot 276 + 3 \cdot 559 + 4 \cdot 801 + 5 \cdot 1108 + \cdots + 11 \cdot 565 + 12 \cdot 290}{10000}$$

$$= 2 \cdot \left(\frac{276}{10000}\right) + 3 \cdot \left(\frac{559}{10000}\right) + 4 \cdot \left(\frac{801}{10000}\right) + 5 \cdot \left(\frac{1108}{10000}\right) + \cdots + 11 \cdot \left(\frac{565}{10000}\right) + 12 \cdot \left(\frac{290}{10000}\right)$$

If simulated an infinite # of times

$$\mu = 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + 5\cdot\frac{4}{36} + \cdots + 11\cdot\frac{2}{36} + 12\cdot\frac{1}{36}$$

$= 7$ (theoretical mean)

In general: $\mu = \sum_{y} y \cdot p(y)$

maximum value $= Y$

$$\mu = \sum_{y=1}^{6} y \cdot p(y) = 1\cdot\frac{1}{36} + 2\cdot\frac{3}{36} + 3\cdot\frac{5}{36} + \cdots 6\cdot\frac{11}{36}$$

$$= 4.47\bar{2}$$

Variance

$$S^2 = (2-\bar{y})^2 \cdot .267 + (3-\bar{y})^2 \cdot .559 + (4-\bar{y})^2 \cdot .801 + \cdots + (11-\bar{y})^2 \cdot .565 + (12-\bar{y})^2 \cdot .290$$

$$\overline{\quad 9999 \quad}$$

$$= (2-\bar{y})^2 \cdot \left(\frac{267}{9999}\right) + (3-\bar{y})^2 \cdot \left(\frac{559}{9999}\right) + \cdots + (12-\bar{y})^2 \left(\frac{290}{9999}\right)$$

An infinite # of trials

$$\sigma^2 = (2-\mu)^2 \cdot \frac{1}{36} + (3-\mu)^2 \cdot \frac{2}{36} + \cdots + (12-\mu)^2 \cdot \frac{1}{36}$$

In general: $\sigma^2 = \sum_y (y-\mu)^2 \cdot p(y)$

Mean: $\mu = \sum_y y \cdot p(y)$

Variance: $\sigma^2 = \sum_y (y-\mu)^2 \cdot p(y)$

Std Dev: $\sigma = \sqrt{\sigma^2}$

Expected Values or Expectations

$$\mu = E(Y) = \sum_y y \cdot p(y)$$

$$E(g(Y)) = \sum_y g(y) \cdot p(y)$$

$$\sigma^2 = E(Y-\mu)^2 = \sum_y (y-\mu)^2 \cdot p(y)$$

Expectation Rules

1. $E(c) = c$

2. $g(Y) = cY.$ $\quad E(g(Y)) = \sum_y cy \cdot p(y)$

$$c\boxed{\sum_y y \cdot p(y)} = c\,E(Y)$$

3. $g(Y) = c_1 g_1(Y) + c_2 g_2(Y) + \cdots + c_k g_k(Y)$

$$E(g(Y)) = E\left[c_1 g_1(Y) + c_2 g_2(Y) + \cdots + c_k g_k(Y)\right]$$

$$= E(c_1 g_1(Y)) + E(c_2 g_2(Y)) + \cdots + E(c_k g_k(Y))$$

4. $$\sigma^2 = \boxed{E(Y-\mu)^2} = E(Y^2) - (E(Y))^2$$

$$E(Y^2) - \mu^2$$

$$E(Y-\mu)^2 = E\left[Y^2 - 2\mu Y + \mu^2\right]$$

$$= E(Y^2) - E(2\mu Y) + E(\mu^2)$$

$$= E(Y^2) - 2\mu \underline{E(Y)} + \mu^2$$

$$= E(Y^2) - 2\mu^2 + \mu^2 = E(Y^2) - \mu^2$$

Sum of 2 Dice

y	
$p(y)$	

$$E(Y) = \mu = 2\left(\tfrac{1}{36}\right) + 3\left(\tfrac{2}{36}\right) + 4\left(\tfrac{3}{36}\right) + \cdots + 11\left(\tfrac{2}{36}\right) + 12\left(\tfrac{1}{36}\right)$$
$$= 7$$

Var(Y)

$$V(Y) = \sigma^2 = (2-7)^2\left(\tfrac{1}{36}\right) + (3-7)^2\left(\tfrac{2}{36}\right) + \cdots + (11-7)^2\left(\tfrac{2}{36}\right) + (12-7)^2\left(\tfrac{1}{36}\right)$$
$$= 5.8\overline{3}$$

$$= E(Y^2) - \mu^2$$
$$= 2^2\left(\tfrac{1}{36}\right) + 3^2\left(\tfrac{2}{36}\right) + 4^2\left(\tfrac{3}{36}\right) + \cdots + 11^2\left(\tfrac{2}{36}\right) + 12^2\left(\tfrac{1}{36}\right) - 7^2$$
$$= 5.8\overline{3}$$

$\boxed{\text{Max}}$

y	1	2	3	4	5	6
$p(y)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$