Gamma Distribution

Note Title 11/5/2008

parameters $\underset{\text{shape}}{\alpha}$ + $\underset{\text{scale}}{\beta}$

$E(Y) = \alpha\beta$ $\qquad E(Y) = \int_0^\infty y \cdot \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}\, dy$

$V(Y) = \alpha\beta^2$

$$= \int_0^\infty \frac{y^{\alpha} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}\, dy = \int_0^\infty \frac{y^{(\alpha+1)-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}\, dy$$

$$\frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \boxed{\frac{y^{(\alpha+1)-1} e^{-y/\beta}}{\beta^{\alpha+1}\Gamma(\alpha+1)}}\, dy$$

gamma pdf with parameter $\alpha+1$, β.

$$= \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha \Gamma(\alpha)} \cdot 1 = \frac{\cancel{\beta^{\alpha+1}}\Gamma(\alpha+1)}{\cancel{\beta^\alpha} \Gamma(\alpha)}$$

$$= \beta \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \beta \cdot \frac{\alpha \cdot \Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta$$

$$E(Y^2) = \int_0^\infty y^2 \cdot \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy$$

$$= \int_0^\infty \frac{y^{(\alpha+2)-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy$$

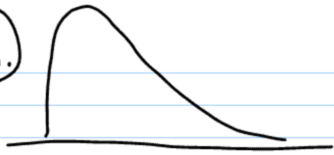
$$= \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{y^{(\alpha+2)-1} e^{-y/\beta}}{\beta^{\alpha+2}\Gamma(\alpha+2)} dy$$

$$= \beta^2 \cdot \frac{(\alpha+1)(\alpha)\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} = \beta^2(\alpha+1)(\alpha)$$

$$E(Y^3) = \beta^3(\alpha+2)(\alpha+1)(\alpha)$$

$$E(Y^4) = \beta^4(\alpha+3)(\alpha+2)(\alpha+1)(\alpha)$$

etc.

(1.) 

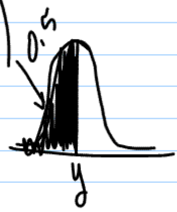
$\alpha = 1.6$
$\beta = 2.0$

(a) $P(Y > 5) = 1 - P(Y < 5) = 1 - F(5)$
$= 1 -$ pgamma(5, shape = 1.6, scale = 2.0)
$= 0.1932$

(b) $E(Y) = \alpha\beta = 1.6(2) = 3.2$ (c) $V(Y) = \alpha\beta^2 = 1.6(2^2) = 6.4$

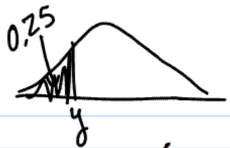
(2.) $\alpha = 1000$ $\beta = 20$

(a) $E(Y) = \alpha\beta = 1000(20) = 20000$
$V(Y) = \alpha\beta^2 = 1000(20)^2 = 400000$

(b) 

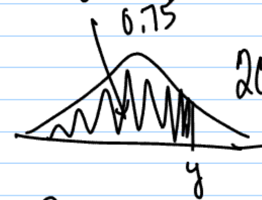
qgamma(0.5, shape = 1000, scale = 20)
$y = 19993.33$

(c) qgamma(0.25, ...)



1969.86

(d) qgamma(0.75, ...)



20,422.87

(e) IQR or Interquartile Range

1969.86 to 20,422.87

<u>Exponential Dist.</u> (Gamma $\alpha = 1$)

Shape is always the same – very right skewed

$$F(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-y/\beta} & 0 \le y < \infty \end{cases}$$

$E(Y) = \beta \qquad V(Y) = \beta^2$

R – (1.) Use gamma (shape=1)

(2.) Use exp , 1/β)

(1.) Y = amount used per day. model with exp. dist.

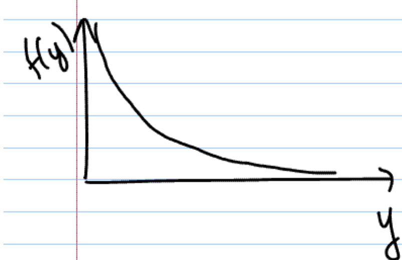
$\beta = 4$

(a) $P(Y>4) = 1 - P(Y \le 4) = 1 - F(4)$

$= 1 - (1 - e^{-4/4}) = e^{-1}$

$= 1 -$ pgamma(4, shape =1, scale =4)

$= 1 -$ pexp(4, 1/4) $= 0.3679$



What will be the median?

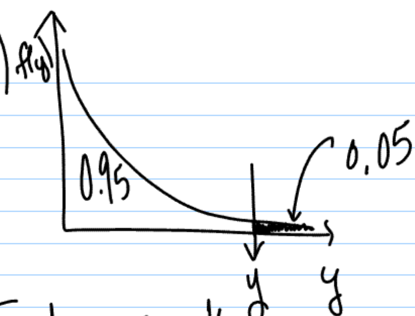
What is y so that $F(y) = 0.5$

$$1 - e^{-y/4} = 0.5$$

$$e^{-y/4} = 0.5 \Leftrightarrow -y/4 = \ln(0.5)$$

$$y = -4\ln(0.5) \quad y = 2.773$$

(b)



Find y so that

$$F(y) = 0.95$$

$$1 - e^{-y/4} = 0.95 \Leftrightarrow y = -4\ln(0.05)$$

qgamma(0.95, shape=1, scale=4)

qexp(0.95, 1/4)

2.

HW

$$E(Y^r) = \int_0^\infty y^r \frac{1}{\beta} e^{-y/\beta}\, dy$$

$$= \int_0^\infty \frac{y^{(r+1)-1} e^{-y/\beta}}{\beta} dy$$

$$= \frac{\beta^{r+1}\Gamma(r+1)}{\beta} \int_0^\infty \frac{y^{(r+1)-1} e^{-y/\beta}}{\beta^{r+1}\Gamma(r+1)} dy$$

$$= \frac{\beta^{r+\cancel{1}}\Gamma(r+1)}{\cancel{\beta}} = \beta^r \Gamma(r+1) = \beta^r r!$$

#9. Y = radii exp $\beta = 10$ ft.

(b) $A = \pi Y^2$

$$E(A) = E(\pi Y^2) = \pi E(Y^2) = \pi \cdot 2\beta^2 = 2\pi\beta^2$$

$$V(A) = V(\pi Y^2) = \pi^2 V(Y^2) = \pi^2\left[E((Y^2)^2) - (E(Y^2))^2\right]$$

$$\pi^2\left[E(Y^4)-(2\beta^2)^2\right]=\pi^2\left[24\beta^4-4\beta^4\right]=\pi^2 20\beta^4$$